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THE APPROXIMATE THEORIES OF PNEUMATIC WAVE GENERATORS

Hydraulic Laboratory Investigation

by

G. H. Keulegan

April 1966

U. S. Army Engineer Waterways Experiment Station
CORPS OF ENGINEERS
Vicksburg, Mississippi

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FOREWORD

Initial authority to conduct a model investigation for the Hilo Harbor Tsunami and Navigation Project was granted by the Office, Chief of Engineers, on 15 September 1961. The U. S. Army Engineer Waterways Experiment Station (WES) was authorized to design a wave machine for the Hilo Harbor tsunami model in a teletype received 16 August 1963 from the U. S. Army Engineer District, Honolulu.

This analytical investigation was conducted by Dr. G. H. Keulegan, consultant, WES Hydraulics Division, simultaneously with tests to investigate design of a bore generator for the Hilo Harbor model, during the period November 1963-May 1964, in the Water Waves Branch, Hydraulics Division, WES, under the direction of Mr. E. P. Fortson, Jr., Chief of the Hydraulics Division, and Mr. R. Y. Hudson, Chief of the Water Waves Branch. The Fortran program for the numerical solution of equations 93, 94, and 95 was worked out by Mr. Michael Dorl of the Hydraulic Analysis Branch. This report was prepared by Dr. Keulegan.

Directors of the WES during the conduct of this investigation and preparation of this report were Col. Alex G. Sutton, Jr., CE, and Col. John R. Oswalt, Jr., CE. Technical Director was Mr. J. B. Tiffany.

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Experience indicates that this condition would be realized for the operation where, subsequent to the raising of water into the chamber, one allows the outside air to enter the chamber through an aperture. Because of the inertia of water, marked pressure oscillations occur in the chamber at the instant of aperture opening. These oscillations last for a short time, and soon the fall of water surface and the increase of pressure, both, are uniform. These aspects of the generator behavior are amenable to analysis.

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GLOSSARY

<u>Pneumatic generator and channel</u>

D Depth of nozzle throat

h Wave height, surge or oscillatory wave

h_m Maximum wave height in oscillatory wave

H Maximum height the water is raised by suction in chamber

H_B Base height of the prismatic portion of channel

H_0 Depth of water in channel downstream of wave front

H_T Height of generator ceiling measured from channel bottom

k Wave number; $k = 2\pi/\lambda$

ℓ Length of pneumatic chamber

ℓ_t Length of nozzle

Q Volume or area of oscillatory wave in channel of unit width, positive or negative portion

t Time

t_0 Characteristic time; see equation 81

T Period

V Value of air gap during wave generation; $V = \ell(\delta + \Delta) \cdot 1$

V_0 Initial value of air gap; $V_0 = \ell\Delta \cdot 1$

δ Fall of water surface in chamber during wave generation

Δ Initial air gap in generator; $H_T = H + \Delta$

λ Wave length

σ Period number; $\sigma - 2\pi/T$

Velocities

q Absolute velocity; $q^2 = u^2 + v^2$

u Horizontal component of velocity or particle velocity in wave motion

u_m Maximum particle velocity in oscillatory wave

u_2 Particle velocity in surges from generator

v Vertical component of velocity

v_n Velocity in the direction of the normal to a surface

v_1 Fall velocity of water surface in chamber

ω Wave velocity in surges

Pressures

A Semiamplitude of the pressure variations

k_1, k_2 Constants to describe the fall of pressure in the pneumatic chamber for surges or long waves; $\Delta p = (1 - k_1 - 2k_2 t)\Delta p_0$

p_0 Atmospheric pressure

P Pressure in chamber measured relative to atmospheric pressure

P_0 Pressure in water at nozzle mouth measured relative to atmospheric pressure

Δp Air suction pressure in chamber during wave generation

Δp_1 Air suction pressure in chamber during the start of waves

Δp_L A positive quantity; represents the additional pressure difference to overcome the friction resistance through the air nozzle

Δp_0 Air suction pressure to raise water to elevation H; $\Delta p_0 = \rho g(H - H_0)$

Energies

E Total energy; $E = T + V$

E_k Internal kinetic energy of air in the vessel

E_ν Rate of dissipation of kinetic energy from the surface of contact

g Constant of gravity

$I\epsilon$ An expression equal to $-\int_0^\epsilon \left(\frac{v_{11}}{v_1} - 1 \right) \frac{\Delta p}{\Delta p_0} \, dt$

M A numerical factor in the expression relating the kinetic energy of liquid in the chamber and the nozzle to h and H_0 ; see equation 33

$\overline{M}$ A numerical factor in the expression relating the kinetic energy of liquid in the chamber and the nozzle to v_1 and ℓ ; see equation 86

N_t A numerical factor in the expression relating the kinetic energy of liquid in the nozzle to u_2 and H_0 ; see equation 26

T Kinetic energy of moving liquids

T_N Kinetic energy in the lower part of the pneumatic chamber

T_2 Total energy, $T_2 = T_{21} + T_{22}$

T_{21} Kinetic energy in ' pneumatic chamber and the nozzle below

T_{22} Kinetic energy in the wave in the channel

V Potential energy of liquids

W Work done by pressures

ΔE Difference in energies of a portion of liquid for the instants t_1 and t_2

Ω Gravitational potential; $\Omega = gy$

Flow of air into receiving vessels

a Cross-sectional area of orifice; $a = \pi d^2/4$

c_0 Velocity of sound in the air outside vessel

C_0, C Initial and later discharge coefficients, respectively, real or apparent, of orifice

C_p Specific heat at constant pressure

C_v Specific heat at constant volume; $C_p = C_v + R$

d Diameter of air orifice

K Numerical defined by $K = \gamma^{1/2} \frac{c_0}{V_0} \left(\frac{p_0}{\Delta p_0} \right)^{1/2}$

m Mass of gas in receiving chamber

M_0 Momentum of liquid entering the orifice

n A constant

N_p Numerical defined by $N_p = \gamma p_0 / \Delta p_0$

p_0 Pressure of outside air

p_1 Pressure at vena contracta of the orifice

p_1' Pressure inside receiving vessel

q_1 Velocity of air at the vena contracta

R Gas constant; $pv = R\theta$

v Specific volume; $v = 1/\rho$

V_j Volume occupied by air jet

V_0 Initial volume of receiving chamber

X_j Length of air jet

β A positive constant

γ Ratio of specific heats; $\gamma = C_p/C_v$

Δ Initial air gap between the water surface and the chamber ceiling

ϵ Internal energy of gas per unit mass; or a numerical which depends on the circumstances of the filling and which can be a variable changing with time

θ Absolute temperature

θ_0 Temperature in degrees Rankine corresponding to 32 F

ρ Density

ρ_0 Density of air outside

ρ_1 Density of air at vena contracta

ρ_1' Density of air inside

σ_1 Cross-sectional area of vena contracta

Geometrical quantities

ℓ,m Directional cosines of normal drawn inward of S

s Length of curve, closed or open

S Surface area

x Longitudinal coordinate

y,z Vertical coordinates

Dimensionless parameters

N $1 + \frac{1}{4}\frac{h}{H_0}$

ζ Nozzle coefficient of resistance; see equation 143

η h/H_0

θ δ/Δ

λ Nozzle factor of resistance; see equation 52

π $\Delta p/\Delta p_0$

τ t/t_0

SUMMARY

After a brief discussion of the genesis of oscillatory waves by the pneumatic method, the required pressure condition for the genesis of long surges with constant wave heights is inquired into. Reasoning from the energy point of view, it is seen that the sum of the pressure head in the pneumatic chamber and the water elevation therein should remain constant during the issuance of the wave from the generator.

Experience indicates that this condition would be realized for the operation where, subsequent to the raising of water into the chamber, one allows the outside air to enter the chamber through an aperture. Because of the inertia of water, marked pressure oscillations occur in the chamber at the instant of aperture opening. These oscillations last for a short time, and soon the fall of water surface and the increase of pressure, both, are uniform. These aspects of the generator behavior are amenable to analysis.

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THE APPROXIMATE THEORIES OF PNEUMATIC WAVE GENERATORS

Hydraulic Laboratory Investigation

PART I: INTRODUCTION

1. The first phase of the Hilo Bay model investigation will comprise examination of the effectiveness of various protective structures against tsunamis using a singular positive wave of translation of great length with large and uniform heights entering the bay. Such a wave can be created in a model by suddenly releasing the impounded waters of a reservoir, by uniformly displacing a bulkhead through a distance, or by using a pneumatic wave generator of considerable capacity. The elementary aspects of the theory relating to the latter will be discussed herein.

2. The original idea of a pneumatic wave generator goes back to the late Professor R. T. Knapp, California Institute of Technology, who used such a machine in a model to study the long wave conditions of Apra Harbor, Guam. Professor Knapp was well known for his ingenious and original ideas in mechanical devices, and the air-operated machine unquestionably is a firm testimonial to his inventive genius. In an unpublished paper[1*] Knapp briefly touched upon the theory of the machine that was used for the Apra Harbor model. In kindly transmitting a copy of the paper to this writer, Mr. J. M. Caldwell of the U. S. Coastal Engineering Research Center added remarks to the effect that there is difficulty in following the arguments of the paper. The brevity of the treatment and possible typographical errors could be the reasons for this difficulty. The explanations of Knapp purport to deal with the machines for the purpose of generating oscillatory translation waves. In the following, before entering into the discussion of air machines to generate singular waves, it would be instructive to consider first the matter of the generator for long oscillatory waves. This is done for the purpose of developing a feeling about the all-round aspects of the problem.

* Raised numerals refer to similarly numbered items in the Literature Cited at end of text.

3. When a pneumatic generator is provided with an exit nozzle, and this seems to be necessary for the genesis of oscillatory shallow-water waves of long length, the flow of water from the pneumatic chamber into the channel outside is one of deflection. Owing to the latter condition, it would appear that the mechanics of flow are more readily treated by resorting to energy considerations rather than to the momentum principle. The energy relation can be expressed in two ways, one representing the Lagrangian point of view and the other the Eulerian point of view. In the review of Knapp's problem, both forms of the energy relation were used. This was successful, and accordingly the same approach was incorporated in the problem of a pneumatic wave generator to produce a singular impulse of wave of great length and of constant height moving over water of constant depth and width.

4. The present report is not meant to be a systematic treatment of pneumatic generators. It merely presents the original thinking that guided the conduct of experiments on a few shapes of generators. The order of the material shown here does represent the progress of analysis running parallel with the tests. The aim was to develop certain basic relations that by themselves were sufficient for design of the generator which could be used in the Hilo Bay model. These relations are shown in the section dealing with design.

5. For the interpretation of the various theoretical results in the paper, reference is made to some of the experimental results with various shapes of pneumatic generators. Test procedures leading to these data are fully explained by Mr. C. C. Shen in a separate report.[2]

PART II: THEORY OF PNEUMATIC GENERATORS FOR OSCILLATORY WAVES

Kinematic Relations

6. Definitions of symbols for wave height and particle velocity (translation waves) are as follows:

- h Wave elevation measured from undisturbed surface
- h_m Semiamplitude of wave height, maximum value
- H_0 Depth of undisturbed water in channel
- T Period
- u Uniform particle velocity in a section
- u_m Semiamplitude of particle velocity, maximum value
- x Distance measured from mouth of nozzle
- λ Wave length

The mouth of the nozzle is that opening where the mean depth of water during a complete oscillation equals H_0. It is assumed that

$$h = h_m \sin (\sigma t - kx); \ \sigma = 2\pi/T$$

and

$$u = u_m \sin (\sigma t - kx); \ k = 2\pi/\lambda$$

The relation between h_m and u_m is determined from the condition of continuity

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uH_0) = 0$$

$$\sigma h_m \cos (\sigma t - kx) - u_m H_0 k \cos (\sigma t - kx) = 0$$

$$u_m = \frac{\sigma}{k}\frac{h_m}{H_0}$$

or

$$u_m = \frac{\lambda}{T}\frac{h_m}{H_0} = \sqrt{gH_0}\,\frac{h_m}{H_0} \tag{1}$$

7. Let Q be the area of the elevated portion of the wave or the depressed portion in channel of unit width. See fig. 1a. For $t = 0$

$$h = h_m \sin kx$$

$$Q = -h_m \int_{\lambda/2}^{\lambda} \sin kx \, dx$$

$$Q = \frac{2h_m}{k} = \frac{h_m \lambda}{\pi} \tag{2}$$

8. Under the action of sinusoidally varying pressure P, the surface of water in the pneumatic chamber during a complete cycle is forced from level A to level B and then is pulled back to the initial level A. The average level is in the same plane as the undisturbed water surface in the outside channel. Let the displacement δ of the water surface in the chamber be measured from the undisturbed level. By storage conditions, where ℓ is the chamber length,

$$\ell \frac{d\delta}{dt} = -uH_0$$

where u is now the particle velocity at the mouth of the nozzle, that is at $x = 0$. As u is sinusoidal, assume that δ is also sinusoidal and, as δ varies from δ_m to $-\delta_m$, then

$$\delta = \delta_m \cos \sigma t$$

At the nozzle mouth, that is, at section $x = 0$, the particle velocities are

$$u = u_m \sin \sigma t$$

This and the storage equation yield

$$2\ell\delta_m = \frac{u_m H_0 T}{\pi} = \frac{h_m \lambda}{\pi}$$

or

$$2\ell\delta_m = Q \tag{3}$$

Accordingly, the pneumatic chamber will require a storage of only Q .

9. To produce waves of specified wave heights $2h_m$, the pressure P of the pneumatic chamber needs to be properly controlled. To determine the relation of P to h_m , resort may be made to the equation of energy to be discussed below.

Energy Equations

10. For the case of two-dimensional flow, the equations of motion are given as

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \Omega}{\partial x} \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial \Omega}{\partial y} \end{aligned} \tag{4}$$

where d/dt denotes differentiation along a particle path, that is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

where u and v are the velocity components in the coordinate axis x and y . The physical meaning of Ω is that it denotes the potential energy, per unit mass, at the point x,y in respect to gravitational forces.[3] As both p , the pressure, and Ω , the gravitational potential, enter into the equation as rates, the resulting motions are not altered if the value of p is reduced by one constant and Ω by another constant. This allows us, when dealing with gravitational wave motion, to put atmospheric pressure equal to zero. The mathematical expression for Ω is gy if y is drawn vertical. Now y may be measured from any horizontal plane. For the problems at hand, y will be measured from the channel bottom.

11. The derivation of the energy equation on the basis of flow equations is given by Lamb.[3] For incompressible flow in two

dimensions, the equation reduces to

$$\frac{d}{dt}(T + V) = \int_s (\ell u + mv)p \, ds \tag{5}$$

which applies to the body of liquid enclosed in a cylinder of unit length in the z direction and having s as the circumferential boundary curve. Here ℓ and m are the directional cosines of an inwardly directed normal to the bounding curve element ds or the bounding surface of area $dA = 1 \cdot s$. The quantities T and V refer to kinetic and potential energies of the liquid bounded by s and are determined from

$$T = \frac{1}{2}\rho \int_S (u^2 + v^2) dS \; ; \; dS = dx \, dy$$

and

$$V = \rho \int_S \Omega \, dS \; ; \; dS = dx \, dy$$

Multiplying the two sides of equation 5 by dt and integrating between two instants t_1 and t_2 yields

$$(T + V)_2 - (T + V)_1 = \int_{t_1}^{t_2} \int_s (\ell u + mv)p \, ds \, dt \tag{6}$$

The interpretation is that the total increase in energy, potential and kinetic, of the same portion of liquid in its path of displacement is equal to the work done by the pressures on its surface. This is stated by Lamb. See fig. 2a. Since the expression in equation 6 gives the change in the energies of the same portion of liquid in its path of displacement for the times t_1 and t_2, this expression may be referred to as the Lagrangian form of the equation of energy.

12. An alternate form of the equation of energy is

$$\frac{\rho}{2} \int_S \frac{\partial q^2}{\partial t} dS = \int_s (\ell u + mv) \left(\frac{\rho}{2} q^2 + \rho\Omega + p \right) ds \tag{7}$$

where $q^2 = u^2 + v^2$ and s represents a curve drawn in the field of velocity and S is the area enclosed by the curve. The interpretation is that the change in kinetic energy in a region delineated in the flow field

by a closed curve is due to the inflow and outflow of potential energy and of kinetic energy into and away from the enclosed area, and to the work done by the pressures on the body of liquid in the delineated area. See fig. 2b. Since the expression in equation 7 gives the changes in the energies in the fixed area, this expression may be referred to as the Eulerian form of the equation of energy. The proof of the expression in equation 7 will be given in the Addendum.

13. In the ordinary application of the energy relations, usually the flow is through passages with rigid walls, for which at the walls u and v vanish.

Pneumatic Pressures

14. Use may now be made of equation 6 to examine the increase in wave energy with time in the channel downstream of the nozzle mouth. See fig. 1b. It is assumed that at time $t < 0$ the liquid in the channel is at rest; the wave motion commences at time $t = 0$; and during time $t = t$ the front, a node point, has traversed the distance $x = x_0$. As

$$kx_0 = \sigma t \tag{8}$$

the disposition of the surface of the wave is

$$h = h_m \sin (kx_0 - kx) \; ; \; 0 < x < x_0$$

and

$$h = 0 \; ; \; x > x_0 \tag{9}$$

15. Consider the area under the wave bounded by the wave surface, the vertical A'B' at $x = 0$, the vertical CD at $x = x_0$, and the bottom segment B'D . Call this the final area S_2 . The liquid of this area initially, that is at time $t = 0$, was contained in the area of the rectangle bounded by the vertical AB at $x = -\Delta x$, the straight line AC , the vertical CD at $x = x_0$, and the bottom segment BD . Call this the initial area S_1 . Since the liquid is incompressible, S_1 and S_2 are equal and

$$H_0 \Delta x = \int_0^{x_0} h \, dx$$

The potential energy in S_2, when referred to the channel bottom, is

$$V_2 = \rho g \int_0^{x_0} \left(\frac{h^2}{2} + H_0 h + \frac{H_0^2}{2} \right) dx$$

the kinetic energy, since $u^2 = gh^2/H_0$, is

$$K_2 = \rho \int_0^{x_0} \frac{u^2 H_0}{2} \, dx = \rho g \int_0^{x_0} \frac{h^2}{2} \, dx$$

and the total energy is

$$(V + K)_2 = \rho g \int_0^{x_0} \left(h^2 + hH_0 + \frac{H_0^2}{2} \right) dx$$

In S_1, the potential energy is

$$V_1 = \rho g \frac{H_0^2}{2} (x_0 + \Delta x)$$

or

$$V_1 = \rho g \int_0^{x_0} \left(\frac{H_0 h}{2} + \frac{H_0^2}{2} \right) dx$$

and, as $u = 0$ for the instant $t = 0$, the kinetic energy K_1 is zero. Denoting the increase in the energies by

$$E_2 = (V + K)_2 - (V + K)_1$$

from the above

$$E_2 = \rho g \int_0^{x_0} \left(h^2 + \frac{1}{2} hH_0 \right) dx$$

After substituting h from equation 9, integration gives

$$E_2 = \frac{\rho g h_m H_0}{2k} (\cos kx_0 - 1) + \frac{\rho g h_m^2}{2k} (kx_0 - \sin kx_0 \cos kx_0)$$

and as $kx_0 = \sigma t$, also

$$E_2 = \frac{\rho g h_m H_0}{2k} (\cos \sigma t - 1) + \frac{\rho g h_m^2}{2k} (\sigma t - \sin \sigma t \cos \sigma t) \tag{10}$$

16. This energy comes from the work done by the pressure P_0 , prevailing at the section $x = 0$, on the line AB , and the work is

$$E_1 = \int_0^t \int_0^{H_0} P_0 u \, dy \, dt$$

The integral E_1 will equal E_2 only if

$$P_0 = \rho g h + \rho g H_0 - \rho g y = \rho g (H_0 + h - y) \tag{11}$$

where h is the surface displacement at $x = 0$. Recalling that $u = u_m \sin \sigma t$ and $h = h_m \sin \sigma t$, the integral for E_1 yields

$$E_1 = \rho g \int_0^t \left(\frac{H_0^2}{2} + hH_0 \right) u \, dt$$

$$= \frac{\rho g h_m H_0}{2\sigma} (\cos \sigma t - 1) + \frac{\rho g h_m u_m H_0}{2\sigma} (\sigma t - \sin \sigma t \cos \sigma t)$$

As $u_m k H_0 = \sigma h_m$, see equation 1, also

$$E_1 = \frac{\rho g h_m H_0}{2k} (\cos \sigma t - 1) + \frac{\rho g h_m^2}{2k} (\sigma t - \sin \sigma t \cos \sigma t) \tag{12}$$

The equality between E_1 and E_2 , thus, is shown.

17. One recognizes, of course, that the pressure P_0 prevailing at the nozzle mouth (that is, at section $x = 0$) is made up of two parts, one due to the wave disturbance h and the other due to the undisturbed liquid of depth H_0 . Having ascertained the magnitude of P_0 , equation 11, the proper pressure P in the pneumatic chamber over the water surface may next be evaluated after resorting to the alternate, or the Eulerian, form of the equation of energy, equation 7. For the present case (see fig. 1c), the bounding curve S is made of the following parts: the segment S_1

consisting of the free surface of the liquid in the chamber, the segment S_2 to be identified as the nozzle mouth, and the rigid boundaries of the chamber and the nozzle in contact with water. The appropriate form of the energy equation, then, is

$$\int_A \frac{\rho}{2} \frac{\partial q^2}{\partial t} dx\ dy = \int_0^{\ell} v_1 \left(\frac{\rho}{2} v_1^2 + \rho\Omega_1 + P \right) d\ell - \int_0^{H} u_2 \left(\frac{\rho}{2} u_2^2 + \rho\Omega_2 + P_0 \right) dy \quad (7a)$$

where v_1 denotes the water surface velocity in the chamber and u_2 the velocity at the mouth. The sums of the terms in the parentheses of the individual integrals are constant, and since

$$\int_0^{\ell} v_1 d\ell = \int_0^{H_0} u_2 dy$$

or

$$v_1 \ell = u_2 H_0$$

the simplified form of the energy equation is

$$\int \frac{\rho}{2} \frac{\partial q^2}{\partial t} dx\ dy + \int_0^{H_0} u_2 dy = \left(\frac{\rho}{2} v_1^2 + \rho\Omega_1 + P \right) - \left(\frac{\rho}{2} u_2^2 + \rho\Omega_2 + P_0 \right) \quad (7b)$$

18. For the purpose of evaluating the term on the left-hand side, measure first the vertical distances in terms of H_0 , and the horizontal distances in the direction of wave motion in terms of $\lambda/2\pi$. Thus, introducing

$$x' = x/H \quad \text{and} \quad y' = 2\pi y/\lambda = y/k$$

the velocity q now is expressed as the function

$$q = u_m f(x', y') \sin \sigma t$$

and with it the left-hand term in equation 7b may be evaluated, giving

$$\int_A \frac{\rho}{2}\frac{\partial q^2}{\partial t}\,dx\,dy + \int_0^{H_0} u_2 dy = N_t \frac{\rho}{2}\frac{\sigma}{k} u_m \cos\sigma t = N_t \frac{\rho}{2} g h_m \cos\sigma t \qquad (13)$$

where N_t is a pure number, its value depending on the shape of the nozzle. Using this result and next expressing v_1 in terms of $u_2 = (u_m \sin \tau)$, the energy relation equation 7b simplifies to

$$\frac{\rho}{2}\left\{N_t g h_m \cos\sigma t + u_m^2\left[1 - \left(\frac{H_0}{\ell}\right)^2 \sin^2 \sigma t\right]\right\} = \rho\Omega_1 - \rho\Omega_2 + P - P_0$$

Introducing into the right-hand member the value of P_0 from equation 11,

$$\Omega_1 = g(H_0 + \delta)\ ,\ \Omega_2 = gv$$

and also writing

$$P = \rho g h_1 - \rho g \delta \qquad (14)$$

the final form of the energy becomes

$$gh_1 = gh_2 + \frac{1}{2} N_t g h_m \cos\sigma t + \frac{1}{2} u_m^2\left[1 - \left(\frac{H_0}{\ell}\right)^2\right]\sin^2 \sigma t$$

Since h_m is small in comparison with H_0, the last term on the right-hand side involving u_m^2 will be small in comparison with gh and, thus, will be ignored. One now has

$$h_1 = h + \frac{1}{2} N_t h_m$$

and accordingly the desired pneumatic chamber pressure should be

$$\frac{P}{\rho g} = h_m \sin\sigma t - \left(\delta_m - \frac{N_t}{2} h_m\right)\cos\sigma t \qquad (15)$$

This may be put in the form

$$\frac{P}{\rho g} = A \sin(\sigma t - \sigma\epsilon) \qquad (16)$$

where

$$A = h_m \sqrt{1 + \left(\frac{N_t}{2} + \frac{\delta_m}{h_m}\right)^2}$$

and

$$\tan \epsilon = - \frac{N_t}{2} + \frac{\delta_m}{h_m}$$

the value of N_t being evaluated from equation 13.

19. One may now summarize the results. The generation of long waves of wave height $2h_m$ and of period T will be brought about in a practical manner through the introduction of sinusoidally varying pressure P in the pneumatic chamber of period T and of semiamplitude ρgA. The phase difference in the pressure variations inside and the wave surface oscillations outside is given by ϵ. Pressure P denotes the excess or deficiency in reference to atmospheric pressure.

20. In the analysis given above and leading to the chamber pressures, losses in th chamber and nozzle passages are ignored. The theoretical treatment of the losses would certainly be somewhat uncertain for nozzles of arbitrary shape. To account for the losses, one may increase the semiamplitude of the pressure variations A by an amount δA. If, in the nozzle used, the two sides of the throat are not symmetrical, losses would be of different amounts for the inflow into the chamber and the outflow. As a result, waves generated in the proximity of the chamber would contain higher harmonics. Perhaps this difficulty can be overcome by choosing a nozzle having small losses. This is a matter that could be examined in models.

21. Simultaneously with the development of thought and the design of pneumatic wave gener[illegible]s at California Institute f Technology, similar developments were c[illegible]ed at the David Taylor Model Basin leading to elaborate machines for the generation of deep-water waves.[4] In the latter type machines the diffuser nozzles are avoided. Under the direction of Mr. Brownell, a system of generators with accessory controls has been evolved which is capable of simulating a composite sea state of any specified complexity. The design criteria of the machines were established from models.

22. In these types of machines with nozzles absent, if desired, the mechanics of flow may be adequately described with results from the momentum principle. A form of the analysis along these lines is given by Kergoat in a brief communication.[5] In the paper are included experimental results which show good agreement with theory.

PART III: APPROXIMATE THEORY OF PNEUMATIC GENERATORS FOR SURGES

23. The feasibility of pneumatic wave generators for the purpose of creating a singular positive translation wave of great length was being examined experimentally. In this section an approximate theory of the generator is developed to serve as a guide for the analysis of the test data.

Wave Characteristics

24. It is desired to produce an elongated wave of constant height h advancing with a constant velocity of propagation ω in still water of constant depth H_0. Applying the Boussinesq concept of propagation to the present case

$$\omega = \sqrt{gH_0}\left(1 + \frac{3}{4}\frac{h}{H_0}\right) \tag{17}$$

since $\partial^2 h/\partial x^2$ vanishes.[6] Let u_2 be the particle velocity in the wave, constant in any vertical section. By the continuity condition

$$u_2(H_0 + h) = \omega h \tag{18}$$

and hence, neglecting powers of h/H_0,

$$u_2 = \omega \frac{h}{H_0}\left(1 - \frac{h}{H_0}\right)$$

Eliminating ω, using equation 17,

$$u_2 = \sqrt{gH_0}\,\frac{h}{H_0}\left(1 - \frac{1}{4}\frac{h}{H_0}\right) \tag{19}$$

Operation of Wave Generator

25. The generator, which consists of an air chamber with a long diffuser type nozzle at the base opening into a channel, is shown in fig. 3a.

This is of the type to be referred to as a low generator. At the top are two apertures, one of them connected to an aspirator and the other to outside air. The openings are controlled. The pressure in the pneumatic chamber is changed from p_0 to $p_0 - \Delta p_0$, where Δp_0 is a positive quantity, by opening the aperture leading to the aspirator. As water in the chamber is raised to a level H, and water remaining in channel is of depth H_0,

$$\rho g H - \Delta p_0 = \rho g H_0$$

or

$$\Delta p_0 = \rho g(H - H_0) \tag{20}$$

At $t = 0$ the aspirator is disconnected, and the aperture to the outside is opened, allowing air to enter. At time t let the chamber relative pressure be $-\Delta p$, δ the fall of water surface in the chamber, L the length of the wave generated, and h the height of the wave with respect to the undisturbed water level.

26. If v_1 is the downward velocity of the falling surface,

$$v_1 = d\delta/dt \tag{21}$$

By the condition of continuity

$$\delta \ell = \int_0^t h \frac{dL}{dt} \cdot dt \tag{22}$$

where ℓ is the longitudinal length of the air chamber. Differentiating with respect to t and since $dL/dt = \omega$

$$v_1 \ell = \omega h \tag{23}$$

Also

$$v_1 \ell = u_2(H_0 - h) \tag{24}$$

<u>Problem of Chamber Pressure</u>

27. From the above developments it is seen that a requisite for the

generation of a wave of constant height h moving with a constant speed of propagation ω in a channel of constant depth H_0 is that the water surface in the pneumatic chamber must fall uniformly (that is, with a constant rate). This poses a restriction on the reduced pressure Δp, and the dependence of $\Delta p/\Delta p_0$ on t may be determined conveniently by again using the principle of energy.

28. In fig. 3a the disposition of liquid in the channel to the left of the limiting section A-A', and in the pneumatic chamber for the initial time $t = 0$, is shown. In fig. 3b the subsequent disposition of the same liquid at time t is shown. The limiting section A-A' is at such a distance from the mouth that the issuing wave has not reached it at the time t. Consider the energies of the water to the left of A-A' for these two instants, $t = 0$ and $t = t$. Since at time $t = 0$ the liquid is at rest, the initial energy of the system is only potential. On the basis of the notations of fig. 3a,

$$(T + V)_1 = \frac{\rho g}{2} (H^2 \ell + H_0^2 L_\infty) \tag{25}$$

At time t the liquid has kinetic energy made of two parts, one (T_{21}) relating to the water in the chamber and in the nozzle, and the other (T_{22}) relating to the wave in the channel. Computation will show that

$$T_{21} = \frac{\rho}{2} v_1^2 (H - \delta - H_B)\ell + \frac{\rho}{2} u_2^2 H_0^2 N_t \tag{26}$$

where N_t is a numerical constant the value of which depends on the shape of the nozzle and the path connecting the main body of water in the chamber with the nozzle. Between the levels of H and H_B the cross section of the pneumatic chamber remains the same. With low generators as in fig. 3, H_B is identical with the narrow opening D of the nozzle. The expression of N_t in terms of nozzle dimensions is derived in the Addendum for two types of pneumatic generators, the low and the elevated. At this time the exact numerical value of N_t is not needed. As regards the second part of the kinetic energy, one has

$$T_{22} = \frac{1}{2} \rho u_2^2 (H_0 + h)L \tag{27}$$

where L is the length of the wave. Finally, the total kinetic energy, $T_2 = T_{21} + T_{22}$, is

$$T_2 = \frac{\rho}{2} v_1^2 (H - \delta - H_B) + \frac{\rho}{2} u_2^2 H_0^2 N_t + \frac{\rho}{2} u_2^2 H_0 \left(1 + \frac{h}{H_0}\right) \tag{28}$$

29. The potential energy of the liquid to the left of A-A' in the chamber and in the channel for the instant t is

$$V_2 = \frac{\rho g}{2} (H - \delta)^2 \ell + \frac{\rho g}{2} (H_0 + h)^2 L + \frac{\rho g}{2} (L_\infty - L) H_0^2 \tag{29}$$

Write $(T + V)_2$ for $T_2 + V_2$. Denote the difference of the energies for the instants $t = t$ and $t = 0$ as

$$\Delta E = (T + V)_2 - (T + V)_1 \tag{30}$$

Substituting from the above and also making use of the continuity condition $\delta \ell = Lh$, since h is constant throughout,

$$\Delta E = -\rho g(H - H_0)hL + \frac{\rho}{2} g\delta hL + \frac{\rho}{2}\left(gh^2 + u_2^2 H_0\right)L$$

$$+ \frac{\rho}{2}\left(u_2^2 - v_1^2\right)hL + \frac{\rho}{2} v_1^2 (H - H_B)\ell + \frac{\rho}{2} u_2^2 H_0^2 N_t \tag{31}$$

For a further simplification of this expression one may first neglect v_1^2 in comparison with u_2^2, since l is many times larger than H_0. Again, two sets of terms may be grouped separately as follows

$$\frac{1}{2}\left(u_2^2 h + gh^2 + u_2^2 H_0\right) = gh^2 N \tag{32}$$

where

$$N = 1 + \frac{1}{4} \frac{h}{H_0}$$

and

$$v_1^2 \, (H - H_B)\ell + u_2^2 H_0^2 N_t = g h H_0^2 M \tag{33}$$

where

$$M = \frac{h}{H_0}\left(1 - \frac{3}{2}\frac{h}{H_0}\right)\left[\frac{H - H_B}{\ell} + \left(1 - 2\frac{h}{H_0}\right)N_t\right]$$

30. The simplified expression for ΔE , equation 31, now is

$$\Delta E = \rho g\left[\frac{1}{2} \, h H_0^2 M + h^2 N L - (H - H_0)hL + \frac{1}{2} \, \delta h L\right] \tag{34}$$

Since $L = \omega t$ and $\delta = v_1 t$, the last may be written also as

$$\Delta E = \rho g\left[\frac{1}{2} \, h H_0^2 M + h^2 N L - (H - H_0)hL + \frac{1}{2} \, \delta h L\right] \tag{35}$$

Here ΔE represents the difference between the energy of the system at time t and that at time $t = 0$. The quantities M and N are independent of time and have set values for H , H_0 , and h constant; and, thus, ΔE is a quadratic algebraic function of time. The variation of Δp with time must be such as to account for this particular form of ΔE . What the desired variation is, as suggested before, may be derived from the energy relation in equation 6. If W is the work done by the pressures on the boundaries of the liquid to the left of A-A' , by the energy principle

$$\Delta E = W \tag{36}$$

where

$$W = -\int_0^{\ell}\int_0^{t} \Delta p v_1 \, d\ell \, dt \tag{37}$$

Since v_1 is constant along the water surface in the pneumatic chamber

$$W = -\ell\int_0^{t} v_1 \Delta p \, dt \tag{38}$$

Before proceeding further let us consider the physical conditions at the instant of wave generation. At $t = 0$, the water in the pneumatic chamber

is at rest. At $t = \epsilon$ and thereafter, water particles in the chamber will be moving with constant velocity v_1, since it is assumed that for $t > \epsilon$ the wave is of constant height. On this basis

$$W = -\ell\int_0^{\epsilon} v_{11}\Delta p \, dt - \ell v_1\int_{\epsilon}^{t} \Delta p \, dt \tag{39}$$

Here v_{11} denotes the surface velocity of water in the chamber for times less than ϵ. We may write also

$$W = -\ell v_1\int_0^{\epsilon} \frac{v_{11} - v_1}{v_1}\Delta p \, dt - \ell v_1\int_0^{t} \Delta p \, dt$$

or

$$W = -\ell v_1 \Delta p_0 \left[\int_0^{\epsilon} \left(\frac{v_{11}}{v_1} - 1\right)\frac{\Delta p}{\Delta p_0} \, dt + \int_0^{t} \frac{\Delta p}{\Delta p_0} \, dt\right]$$

and if we put

$$-I\epsilon = \int_0^{\epsilon} \left(\frac{v_{11}}{v_1} - 1\right)\frac{\Delta p}{\Delta p_0} \, dt \tag{40}$$

$$W = -\ell v_1 \Delta p_0 \left(-I\epsilon + \int_0^{t} \frac{\Delta p}{\Delta p_0} \, dt\right) \tag{41}$$

Using one of the forms of the condition of continuity, equation 24, and introducing the value of Δp_0, equation 20, yields

$$W = -u_2(H_0 + h)\rho g(H - H_0)\left(-I\epsilon + \int_0^{t} \frac{\Delta p}{\Delta p_0} \, dt\right)$$

or

$$W = -\rho g u h(H - H_0)\left(-I\epsilon + \int_0^{t} \frac{\Delta p}{\Delta p_0} \, dt\right) \tag{42}$$

31. For $t > \epsilon$, we suppose that the pressures are given as

$$\frac{\Delta p}{\Delta p_0} = 1 - k_1 - 2k_2 t \tag{43}$$

and if we denote by Δp_i the pressure that would exist at $t = 0$

$$\Delta p_i = (1 - k_1)\Delta p_0 \tag{44}$$

and hence

$$k_i = \left(1 - \frac{\Delta p_i}{\Delta p_0}\right) \tag{45}$$

Now

$$\int_0^t \frac{\Delta p}{\Delta p_0}\, dt = (1 - k_1)t - k_2 t^2$$

and this makes

$$W = -\rho g\omega h(H - H_0)\left[-I\epsilon + (1 - k_1)t - k_2 t^2\right] \tag{46}$$

which like ΔE, equation 35, is an algebraic quadratic function of t. Comparing terms in the two expressions, equations 35 and 46, not involving t one finds

$$\rho g(H - H_0)I\epsilon = \frac{1}{2} h^2 H_0 M \tag{47}$$

Accordingly, since M is a positive number, I is also a positive number. The suggestion is that there should be a relatively sudden increase of pressure which then after a short period of time falls to Δp_i. Further discussion as regards the transient pressure may be omitted here. Initial conditions will be discussed later.

32. Comparing the coefficients of t in the expressions of ΔE and W, equations 35 and 46, one finds

$$hN - (H - H_0) = -(1 - k_1)(H - H_0)$$

or (48)

$$hN = (H - H_0)k_1$$

$$\frac{h}{H_0} = \left(\frac{H}{H_0} - 1\right)\left(1 - \frac{1}{4}\frac{h}{H_0}\right)k_1 \tag{49}$$

This expression enables one to determine k_1 in terms of the wave height h and the elevation of the water surface in the pneumatic chamber H at the time the wave starts. From k_1 one next obtains the value of the pressure in the chamber to generate the wave of the desired height h.

33. Comparing the coefficients of t^2 in the expressions of ΔE and W, equations 35 and 46, one finds that

$$1/2v_1 = (H - H_0)k_2 \tag{50}$$

From the continuity relation

$$v_1 \ell = \omega h = h\sqrt{gH_0}\left(1 + \frac{3}{4}\frac{h}{H_0}\right)$$

The above yields

$$2k_2 = \frac{h\sqrt{gH_0}}{\ell(H - H_0)}\left(1 + \frac{3}{4}\frac{h}{H_0}\right) \tag{51}$$

which determines what the rate of fall of pressure in the pneumatic chamber should be in order to maintain a constant wave height with time in the generated wave. Here k_2 is determined in terms of h, H, and ℓ, the latter being the dimension of the tank in the longitudinal direction.

34. Equation 50 may be written in another form. Since $v_1 = d\delta/dt$ and $2k_2 = -d(\Delta p/\Delta p_0)/dt$,

$$\frac{d\delta}{dt} = (H - H_0)\frac{d}{dt}\left(\frac{\Delta p}{\Delta p_0}\right)$$

But

$$\Delta p_0 = \rho g(H - H_0)$$

and hence

$$\frac{d\delta}{dt} = -\frac{d}{dt}\left(\frac{\Delta p}{\rho g}\right) \tag{50a}$$

indicating that the rate of fall of water surface in the chamber equals the rate of increase of chamber pressure.

35. The relation, equation 49, connecting h with k_1 is for the condition that in the flow of water through the nozzle there is no loss of energy from friction associated with turbulence and deflected motions. Since in the best designed nozzle some loss is expected to occur, a provision must be made to account for the effect of the loss. The frictional loss reduces the energy available to create the wave. Let this loss be ΔL, and write

$$\Delta L = \lambda \rho g h^2 NL \tag{52}$$

as discussed in the Addendum. This means that in the energy expression ΔE, equation 37, in the place of $\rho g h^2 NL$ one must now write $(1 + \lambda)\rho g h^2 NL$. Proceeding as before one now has

$$\frac{h}{H_0} = \left(\frac{H}{H_0} - 1\right)\left(1 - \frac{1}{4}\frac{h}{H_0}\right)\frac{k_1}{1 + \lambda} \tag{53}$$

We may refer to λ as the nozzle friction facto .

36. Examining the two expressions, equations 51 and 53, it is seen that the quantities k_1 and k_2 are not independent but are related. Comparing these equations with each other, it is found that

$$\frac{2k_2}{k_1} = \frac{1}{1 + \lambda}\frac{\sqrt{gH_0}}{\ell}\left(1 + \frac{1}{2}\frac{h}{H_0}\right) \tag{54}$$

so that, if the friction factor is known, the ratio $2k_2/k_1$ can be evaluated from the assumed values of h and H_0 and the chamber size ℓ. Conversely, if $2k_2$ and k_1 are observed, the relation is utilized to compute λ, since

$$1 + \lambda = \frac{k_1}{2k_2}\frac{\sqrt{gH_0}}{\ell}\left(1 + \frac{1}{2}\frac{h}{H_0}\right) \tag{55}$$

Experimental Evidence

37. The preceding analysis dealing with the generation of an elongated wave of constant height suggests that this kind of wave would be more readily and easily generated if the pneumatic chamber is elevated. If the chamber is elevated, the total fall of water surface therein would be a smaller fraction of the initial height H , and accordingly the pressures could be conveniently produced without a variable control of the air opening. To examine the feasibility of the idea, experiments were conducted with a generator as shown in fig. 4. The relations shown in equations 51 and 53 still hold. The only modification would be in regard to the coefficient M , equation 47, which changes with the type of generator adopted. This is not important since transient pressure effects at the time of wave issuance will be ignored.

38. For the present, the important observation data to be examined are: the wave height at the mouth of the nozzle as a function of time, the pressure variation in the chamber, and the fall of water surface inside the chamber. An example of a tracing from an electronic record is shown in fig. 5. The lower curve shows the variation in wave height in the vicinity of the nozzle mouth. Note that for a long time the wave height remains constant. The middle curve shows the increase with time of the pressure in the chamber and in the area above the water. If one ignores the oscillation of pressure in the short duration following the instant that the air passage is opened, the increase of the pressure with time is linear. It would be adequate to represent the pressure course by a single straight line for all instants. Note that the intersection of the pressure line with the zero time axis is indicated by Δp_i . For the present the reason for the pressure oscillations for small t values will not be discussed. The matter will be taken up in another section. The upper curve shows the fall of the water surface in the chamber with time. The rate of fall is constant, and its absolute value equals the rate of rise of the pneumatic pressure. These results are in accordance with theory. The data obtained from this and a few other records are presented in table 1.

39. The data will be examined to compare the theoretical results

with the observed results. The quantities involved are

$$u_2 = \sqrt{gH_0}\,\frac{h}{H_0}\left(1 - \frac{1}{4}\frac{h}{H_0}\right) \qquad (19 \text{ bis})$$

which is the expression for the particle velocity in the wave ;

$$2k_2 = \frac{h\sqrt{gH_0}}{\ell(H - H_0)}\left(1 + \frac{3}{4}\frac{h}{H_0}\right) \qquad (51 \text{ bis})$$

which is the expression for the rate of increase of pressure in the pneumatic tank and time; and

$$\frac{h}{H_0} = \left(\frac{H}{H_0} - 1\right)\left(1 - \frac{1}{4}\frac{h}{H_0}\right)\frac{k_1}{1 + \lambda} \qquad (53 \text{ bis})$$

which is the expression that determines the required (effective!) pressure in the tank at the time when the wave commences to issue from the nozzle.

40. The constants k_1 and λ appear in the last equation, and thus it would serve to determine λ , the nozzle resistance factor, in terms of h and k_1 as observed quantities. Another way to determine λ is from

$$1 + \lambda = \frac{k_1}{2k_2}\,\frac{\sqrt{gH_0}}{\ell}\left(1 + \frac{1}{2}\frac{h}{H_0}\right) \qquad (55 \text{ bis})$$

41. It is helpful to mention that the quantities k_1 and k_2 are not observed but are derived from the observed quantities $\Delta p_0/\rho g$, $\Delta p/\rho g$, and $d/dt \cdot \Delta p/\rho g$, using the definitions

$$2k_2 = -\frac{d}{dt}\,\Delta p/\Delta p_0$$

and

$$k_1 = 1 - \Delta p_1/\Delta p_0 \qquad \text{(See equations 43 and 45.)}$$

42. The computed values are shown in the lower part of table 1. There is a fair degree of agreement between the observed and the computed values of u_2 and k_2 . There are two determinations of λ , and as expected, there are considerable variations from one run to another.

PART IV: THE MECHANICS OF PRESSURE BUILDUP IN PNEUMATIC CHAMBERS

43. It was shown in Part III that the mere raising of the level of the water in the chamber and allowing the outside air to enter through an aperture without the assistance of additional controls is sufficient to cause the pressure to fall at a constant rate. The fall is as required by the theory of long waves of constant height. This behavior of the generator needs to be explained on the basis of the mechanics of air flow into the chamber.

Mass Rate of Air Inflow

44. As a first step in the analysis, consider the closed vessel shown in fig. 6 which has an opening in the form of a circular orifice of area a. Let the volume of the vessel be V_0. When the pressure inside is below atmospheric there is a mass rate of flow inward. Let the conditions outside be given by p_0 and ρ_0 and those of the interior (that is, at the vena contracta) p_1 and ρ_1. Let σ_1 be the area of the vena contracta. Assume an adiabatic flow. For the mass rate of flow one has from Lamb[3]

$$\frac{dm}{dt} = \rho_1\sigma_1 q_1 = \left(\frac{2}{\gamma - 1}\right)^{1/2} c_0\rho_0\left[\left(\frac{p_1}{p_0}\right)^{\frac{2}{\gamma}} - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma+1}{\gamma}}\right]^{1/2} \tag{56}$$

where q_1 is the velocity of air at the vena contracta, c_0 is the velocity of sound in the air outside, that is

$$c_0 = \left(\gamma\,\frac{p_0}{\rho_0}\right)^{1/2} = (\gamma R\theta_0)^{1/2} \tag{57}$$

and γ is the ratio of the specific heats at constant pressure and constant volume. For air $\gamma = 1.408$, and for standard atmosphere $c_0 = 1089$ ft/sec; θ_0 is the temperature in degrees Rankine corresponding to 32 F.

45. In our applications the difference between p_0 and p_1 is small. Denoting the difference by Δp, $p_1/p_0 = 1 - \Delta p/p_0$.

$$\left(\frac{p_1}{p_0}\right)^{\frac{2}{\gamma}} = 1 - \frac{2}{\gamma}\frac{\Delta p}{p_0} ; \left(\frac{p_1}{p_0}\right)^{\frac{\gamma+1}{\gamma}} = 1 - \frac{\gamma + 1}{\gamma}\frac{\Delta p}{p_0}$$

With these approximations the expression of mass rate of flow, equation 56, reduces to

$$\frac{dm}{dt} = \left(\frac{2}{\gamma}\right)^{1/2} c_0\rho_0\sqrt{\frac{\Delta p}{p_0}} \cdot \sigma_1$$

Introducing the value of c_0 from above, equation 57, yields

$$\frac{dm}{dt} = 2^{1/2}\rho_0^{1/2}\sqrt{\Delta p} \cdot \sigma_1$$

and putting

$$2^{1/2}\sigma_1 = Ca \tag{58}$$

finally yields

$$\frac{dm}{dt} = C\rho_0^{1/2}a\sqrt{\Delta p} \tag{59}$$

One may refer to C as the coefficient of discharge, since it is a dimensionless quantity.

46. To coefficient C one may ascribe a meaning more general than that implied by equation 58. Equation 59 may be used also in the cases where there is dissipation in the flow reaching the vena contracta or the passage may be through a tube of any kind. The formula serves to relate the mass flow of air to the pressure in the closed chamber. In such cases C may be thought to depend on Reynolds number $R_e = \sqrt{\frac{\Delta p}{\rho}} \cdot \frac{d}{\nu}$, d being the diameter of the tube.

47. It is advantageous to express the relation in equation 59 in a dimensionless form. Omitting the steps of transformation the expression is

$$\frac{dm}{dt} = Ca\rho_0\frac{c_0}{\gamma^{1/2}}\sqrt{\frac{\Delta p}{p_0}} \tag{59a}$$

To illustrate we now consider the Fliegner formula cited by Prandtl.[7]

Polson et al.[8] have shown the validity of the formula. In metric units

$$Q = 0.76a\sqrt{(p_1 - p_2)p_2/T_1}$$

In the terms of this paper and if Δp is small with respect to p_0,

$$g\frac{dm}{dt} = 0.76a\sqrt{\Delta p_0 \cdot p_0/\theta}$$

One has, omitting the steps of transformation,

$$\frac{dm}{dt} = 0.76a\frac{R^{1/2}}{g}\rho_0\sqrt{\frac{\Delta p}{p_0}} = 1.31\frac{c_0 a\rho_0}{\gamma^{1/2}}\sqrt{\frac{\Delta p}{p_0}}$$

since $R/g = 29.4$ meters per degree. Accordingly, $C = 1.31$ and the vena contracta occupies the entire area of the opening.

Determination of Coefficient of Discharge

48. In the applications to be made eventually, it will be necessary to know the value of the discharge coefficient. A method to determine it would be to ascertain the rate of increase of the pressure inside the vessel with an initial difference of the pressure Δp_0. If the volume V_0 is large enough the filling may be regarded as an adiabatic process. Then, the pressure and density in the vessel are those at the vena contracta.

49. In this hypothesis one uses

$$m = \rho_1 V_0$$

and

$$\frac{dm}{dt} = \frac{d\rho_1}{dt}V_0 \tag{60}$$

since V_0 remains constant. Now for adiabatic conditions

$$\frac{p_0 - \Delta p}{p_0} = \left(\frac{\rho_1}{\rho_0}\right)^{\gamma} \tag{61}$$

or

$$\left(\frac{p_0 - \Delta p}{p_0}\right)^{\frac{1}{\gamma}} = \frac{\rho_1}{\rho_0}$$

Since Δp is a small quantity

$$\frac{\rho_1}{\rho_0} = 1 - \frac{1}{\gamma}\frac{\Delta p}{p_0}$$

and differentiating with respect to time

$$\frac{p_0}{\rho_0}\frac{d\rho_1}{dt} = -\frac{1}{\gamma}\frac{d\Delta p}{dt}$$

Introducing the sound velocity c_0 , $c_0^2 = \gamma p_0/\rho_0$

$$\frac{d\rho_1}{dt} = -\frac{1}{c_0^2}\frac{d\Delta p}{dt}$$

and from equation 60

$$\frac{dm}{dt} = -\frac{V_0}{c_0^2}\frac{d\Delta p}{dt}$$

Equating this to the mass rate of entry from outside, equation 59,

$$\frac{V_0}{c_0^2}\frac{d}{dt}\Delta p = -\rho_0^{1/2}\, Ca \sqrt{\Delta p}$$

or

$$\frac{d}{dt}\frac{\Delta p}{\Delta p_0} = -\frac{c_0^2}{V_0}\frac{\rho_0^{1/2}}{\Delta p_0^{1/2}}\, Ca \sqrt{\frac{\Delta p}{\Delta p_0}}$$

or

$$\frac{d}{dt}\frac{\Delta p}{\Delta p_0} = -KCa \sqrt{\frac{\Delta p}{\Delta p_0}} \qquad (62)$$

where

$$K = \gamma^{1/2}\frac{c_0}{V_0}\left(\frac{p_0}{\Delta p_0}\right)^{1/2}$$

The solution is

$$\sqrt{\frac{\Delta p}{\Delta p_0}} = 1 - \frac{KCa}{2} t \qquad (63)$$

50. An experimental determination of C was carried out by noting the filling rates of a rectangular box 2 ft high and with a cross section of 1 by 2 ft. In the upper plate (0.485 in. thick) numerous circular apertures with varying diameters were drilled. The edges of the apertures were sharp. Taking an aperture at a time when the air pressure in the vessel was reduced by an amount Δp_0, the aperture was next opened to the air outside, and the increase of the pressure within the vessel was noted. Fig. 7 is an example of a plot showing the linear variation of $(\Delta p/\Delta p_0)^{1/2}$ with time. This is in agreement with the theoretical form in equation 63. Thus, if m is the slope, the coefficient is

$$C = 2m/Ka$$

The values of C for orifices of various diameters are shown in table 2. The coefficient of discharge is practically independent of the diameters except that the coefficient shows a slight increase in value when the ratio of plate thickness to orifice diameter is increased. The constant value is $C = 0.71$.

51. The tests were continued with vessel volumes V_0 smaller than 4.03 ft. These lesser volumes were conveniently obtained by filling the original vessel with water. The results of the tests are shown in table 3. It is seen that C when determined by equation 63 decreases with V_0. In the routine tests on wave generation it was observed that owing to inertia the water surface in the chamber remains still for a short time following the instant when the chamber is opened to outside air. This would mean that for the initial period the air is flowing into a closed vessel of unvarying volume. In figs. 8 and 9 the initial increase of pressure with time for the two types of generators is shown. The air openings were in the form of circular orifices. Since the manner of the increase is in agreement with equation 63, C may be computed by this formula. The

results of the reductions are shown in table 4.

52. It will be shown in the Addendum that the effective values of C (that is, the values computed on the basis of equation 63) may be dependent on the volume V_j occupied by the air jet and the length X_j of the jet, provided that a short distance below the air opening the jet is deflected laterally over the surface of the water. This would suggest that C could be a function of V_j/V_0 and X_j/Δ where Δ is the initial air gap between the water surface and the chamber ceiling. In fig. 10 the effective discharge coefficient is plotted against X_j/Δ. There is a fair congruency of the points with the curve drawn. For the cases considered, the effect of the ratio V_j/V_0 on C is negligible. The meanings of X_j and V_j are discussed in the Addendum.

53. In a more general sense, as previously mentioned, C will be interpreted to be a proportionate factor in the empirical formula, equation 59, connecting the mass rate of flow of air with the pressure in a receiving chamber. The air passage may be of any form. In the first runs of pneumatic wave generation the chambers were connected to the aspirator and to the air outside by means of a so-called three-way valve. In fig. 11 are shown the initial rises of pressures and these also yield smaller discharge coefficients.

54. The present study on the discharge coefficients unfortunately is very incomplete. It would have been the better procedure to base the evaluation of C on the direct measurement of the mass rate of flow as determined by the simultaneous temperature and pressure measurements in the receiving vessel as function of time. It was the intention to follow the procedure in an extended study, but due to the lack of a sensitive transistor for the temperature measurements at that time, the idea was abandoned. Another precaution in the conduct of tests would have been the measurements of the pressures in various locales, one important place being the vicinity of the orifice. There was only one pressure-measuring element used in the tests made, and this definitely was not sufficient. Again, for a more reliable control of the tests, it would have been desirable to connect the receiving vessel to a very large discharging vessel to replace the outside air. In this arrangement there would be an additional means to

determine the mass rate of flow of the air.

55. The main assumption of the above analysis leading to equation 63 was that

$$\frac{\rho_1}{\rho_0} = 1 - \frac{1}{\gamma}\frac{\Delta p}{p_0}$$

Now suppose that in an actual situation the relation between the pressure and the density of the air in the receiving vessel is

$$\frac{\rho_1}{\rho_0} = 1 - \frac{\epsilon}{\gamma}\frac{\Delta p}{p_0}$$

where ϵ is a numerical, the value depending on the circumstances of the filling; it can be a variable, changing with time. If the process is to approximate isothermal adjustment, it may be a number close to γ. Repeating the analysis one now obtains, in the place of equation 62, the relation

$$\frac{d}{dt}\left(\epsilon \frac{\Delta p}{\Delta p_0}\right) = -KCa\sqrt{\frac{\Delta p}{\Delta p_0}} \qquad (64)$$

where K has the same meaning as before. When information relating to ϵ is lacking, owing to the insufficiency of observation, the solution of equation 64 is uncertain.

56. In any case one may revert to equation 62 or to equation 63 as a convention, and C determined in this manner would be referred to as the effective coefficient of discharge.

Pressure Buildup in Pneumatic Chambers

57. In computing the pressure changes in pneumatic chambers open to air, due consideration must be given to the fact that during the generation of an elongated wave of constant depth the air volume in the chamber would be changing. The change is measured in terms of the fall of the water surface inside. Ideally, the fall must be uniform; that is, if V is the air volume at time t and V_0 the initial value

$$V/V_0 = 1 + nt \tag{65}$$

where n is a constant. Simultaneously the pressure buildup in the chamber must conform to the expression

$$\Delta p/\Delta p_0 = 1 - k_1 - 2k_2 t \tag{43 bis}$$

especially in the later stages of the wave formation. The bearing of the quantities k_1 and k_2 upon the wave height h and the water depth in the channel H_0 has been shown previously through equations 51 and 53.

58. The rate n here is proportional to $2k_2$. Denoting the elevation of the chamber ceiling from the channel bottom by H_T

$$V_0 = (H_T - H)\ell$$

and

$$V = (H_T - H + \delta)\ell$$

Hence,

$$\frac{V}{V_0} = 1 + \frac{\delta}{\Delta} \tag{66}$$

where

$$\Delta = H_T - H \tag{67}$$

Comparing this with equation 65

$$n = \frac{1}{\Delta}\frac{d\delta}{dt} \tag{68}$$

or

$$n = v_1/\Delta$$

and from equation 50

$$n = \frac{H - H_0}{\Delta} \cdot 2k_2$$

or

$$n = \frac{H - H_0}{H_T - H} \cdot 2k_2 \tag{69}$$

59. In the experiments of pneumatic wave generators of the type

considered here, the passage from the pneumatic chamber to the outside air consisted either of a short tube or a rubber hose leading from the short tube or a circular orifice. In all these cases, once the water was sucked to level H and the passage was opened to air outside, the fall of the water surface inside and the subsequent increase of pressure in the chamber took place uniformly without further control of the air passage. This matter deserves an analytical examination, and it will be supposed that the air aperture is in the form of an orifice, since the coefficient of discharge of this type aperture is known.

60. We take the general problem of flow of air into a vessel of varying internal volume. The mass of air in the chamber is $V\rho_1$ and using equation 66

$$m = V_0\left(1 + \frac{\delta}{\Delta}\right)\rho_1 \tag{70}$$

Differentiating with respect to time

$$\frac{dm}{dt} = V_0\left(1 + \frac{\delta}{\Delta}\right)\frac{d\rho_1}{dt} + \frac{V_0\rho_1}{\Delta}\frac{d\delta}{dt}$$

or

$$\frac{dm}{dt} = V_0\rho_0\left[\left(1 + \frac{\delta}{\Delta}\right)\frac{d}{dt}\frac{\rho_1}{\rho_0} + \frac{\rho_1}{\rho_0}\frac{1}{\Delta}\frac{d\delta}{dt}\right] \tag{71}$$

Assuming an adiabatic process and remembering that the pressure inside the chamber differs but little from the outside atmospheric pressure,

$$\frac{\rho_1}{\rho_0} = 1 - \frac{1}{\gamma}\frac{\Delta p}{p_0}$$

Substituting this information in equation 70,

$$\frac{dm}{dt} = \rho_0 V_0\left[-\left(1 + \frac{\delta}{\Delta}\right)\frac{1}{\gamma p_0}\frac{d\Delta p}{dt} + \left(1 - \frac{1}{\gamma}\frac{\Delta p}{\Delta p_0}\right)\frac{1}{\Delta}\frac{d\delta}{dt}\right]$$

or

$$\frac{dm}{dt} = \rho_0 V_0\frac{\Delta p_0}{\gamma p_0}\left[-\left(1 + \frac{\delta}{\Delta}\right)\frac{d}{dt}\frac{\Delta p}{\Delta p_0} + \left(\frac{\gamma p_0}{\Delta p_0} - \frac{\Delta p}{\Delta p_0}\right)\frac{1}{\Delta}\frac{d\delta}{dt}\right]$$

Since Δp is small in comparison with γp_0 ; the latter equation may be simplified, ignoring $\Delta p/\Delta p_0$, to

$$\frac{dm}{dt} = \rho_0 V_0 \frac{\Delta p_0}{\gamma p_0} \left[- \left(1 + \frac{\delta}{\Delta}\right) \frac{d}{dt} \frac{\Delta p}{\Delta p_0} + N_p \frac{1}{\Delta} \frac{d\delta}{dt} \right] \tag{72}$$

where

$$N_p = \gamma \frac{p_0}{\Delta p_0} \tag{73}$$

61. It will be recalled that the mass rate of air flow from outside into a vessel will be expressed by

$$\frac{dm}{dt} = \rho_0^{1/2} \, Ca(\Delta p_0)^{1/2} \sqrt{\Delta p/\Delta p_0} \tag{59 bis}$$

or writing

$$\pi = \Delta p/\Delta p_0$$

$$\frac{dm}{dt} = \rho_0^{1/2} \, Ca(\Delta p_0)^{1/2} \, \pi^{1/2}$$

Here, C and a are the coefficient of discharge and the area of the orifice, respectively. Eliminating dm/dt between equations 59 bis and 72 and making some simple transformations, the result is

$$-\left(1 + \frac{\delta}{\Delta}\right) \frac{d\pi}{dt} + N_p \frac{d}{dt} \frac{\delta}{\Delta} = KCa\pi^{1/2} \tag{74}$$

where

$$N_p = \gamma \frac{p_0}{\Delta p_0} \quad \text{and} \quad K = \frac{\gamma^{1/2} c_0}{V_0} \left(\frac{p_0}{\Delta p_0}\right)^{1/2}$$

This is the general equation for the buildup of the pressure in a vessel whose internal volume $V = V_0(1 + \delta/\Delta)$ changes with time. To specialize, put

$$\delta/\Delta = nt$$

and introduce this in equation 74; since n will be taken to be a constant, the result is

$$-(1 + nt)\frac{d\pi}{dt} + N_p n = KCa\pi^{1/2} \qquad (75)$$

The dimensions of the product KCa are the inverse of time. Introduce

$$t_0 = (KCa)^{-1}$$

$$\tau = t/t_0$$

$$m = n \cdot t_0$$

With these, equation 75 takes the dimensionless form

$$-(1 + m\tau)\frac{d\pi}{d\tau} + mN_p = \pi^{1/2} \qquad (76)$$

This is the differential equation that should apply for the pressure build-up in the pneumatic chamber when the water surface is falling uniformly. The differential equation with m a constant permits an exact solution. For the purpose at hand, however, it would be better to consider a series solution in the ascending powers of τ .

$$\pi = \pi_0 + \left(\frac{d\pi}{d\tau}\right)_0 \tau + \frac{1}{2}\left(\frac{d^2\pi}{d\tau^2}\right)\tau^2 + \ldots \qquad (77)$$

where the 0 subscript outside of a parenthesis indicates that the quantity is to be evaluated for $\tau = 0$. Assuming that C is independent of time, and this would be the case if the coefficient of discharge were not affected by the volume of air in the pneumatic chamber, one will obtain the following sequence for the coefficients appearing in equation 77 by referring to equation 76 and those obtained by the successive differentiation of the latter with time. The sequence is:

$$\left.\begin{aligned} \pi_0 &= 1 - k_1 \\ \left(\frac{d\pi}{d\tau}\right)_0 &= mN_p - \pi_0^{1/2} \\ -\left(\frac{d^2\pi}{d\tau^2}\right)_0 &= \left(m + \frac{1}{2}\pi^{1/2}\right)\left(\frac{d\pi}{d\tau}\right)_0 \end{aligned}\right\} \qquad (78)$$

The quantities in the first parenthetical term on the right-hand side of the last equation are positive, and therefore the second time derivative of π does not vanish. The meaning is that the curve of π versus time is not a straight line but is curved slightly. The solution of equation 76 does not exactly reproduce the linear variation of the pressure in accordance with equation 43. It is, however, expected that the time variation of π as given by the second line in equation 78 would be sufficiently correct. Identifying $(d\pi/d\tau)_0$ with $-2k_2t_0$ and supposing that k_1 is small in comparison with unity, the second equation in equation 78, after reverting to the original variables, yields

$$2k_2 = N_p n - KCa$$

Introducing n from equation 69, remembering that $N_p = \gamma p_0/\Delta p_0$, and simplifying, one has

$$2k_2 = \frac{KCa}{1 + \gamma \dfrac{H - H_0}{H_T - H} \cdot \dfrac{p_0}{\Delta p_0}}$$

Since $p_0/\Delta p_0$ is a large number, the above may be written next as

$$2k_2 = \frac{KCa}{\gamma \dfrac{H - H_0}{H_T - H_0} \cdot \dfrac{p_0}{\Delta p_0}} \qquad (79)$$

where

$$K = \frac{c_0\gamma^{1/2}}{V_0}\left(\frac{p_0}{\Delta p_0}\right)^{1/2}$$

Equation 79 purports to connect the rate of increase of air pressure in the pneumatic chamber with the area of the orifice opening. From the tests on the orifices one suspects that C is close to unity. Fairly reliable values of C may be obtained, it is believed, on the basis of equation 79, using data of the quantities appearing in this equation. The pertinent data are shown in table 5, together with the computed values of C.

For the four tests the mean value of C is 0.95.

62. Because of the manner of derivation the values of C just referred to apply for small τ or small water displacement in the pneumatic chamber. What the values may be for subsequent times or greater displacement can be obtained from equation 74 inserting in it the observed values of $d\pi/dt$, $d\delta/dt$, π , and δ . The values of C that were evaluated in this manner are presented in table 6. It was mentioned previously that during the action of the pneumatic generators the water in the chamber remains immobile for a few seconds after the orifice is opened to outside air. Thus, for this initial period $d\delta/dt$ is zero. Accordingly, a specific value C may be computed from equation 74, putting $d\delta/dt$ equal to zero. Values thus obtained, see figs. 8 and 9, are also included in table 6. Now C is a discontinuous function of τ or θ , so that if equation 74 is chosen as a basis for the analytical treatment of generator action, this discontinuity must be borne in mind. This may be accomplished in the following manner. Let C_0 be the effective coefficient for the initial period or for the times when $d\delta/dt$ vanishes and C the effective coefficient for the later times. Denote the ratio of C to C_0 by r

$$C = rC_0$$

Insert this in equation 74 and write in the resulting equation

$$\theta = \delta/\Delta$$

This gives

$$-(1 + \theta)\frac{d\pi}{dt} + N_p \frac{d\theta}{dt} = KC_0 a r \pi^{1/2} \tag{80}$$

Note that the dimensions of KC_0a are that of time inverse; accordingly, put

$$t_0 = (KCa)^{-1} \tag{81}$$

and introduce the dimensionless time variable

$$\tau = t/t_0$$

in the above equation. The result is

$$-(1 + \theta) \frac{d\pi}{d\tau} + N_p \frac{d\theta}{d\tau} = r\pi^{1/2} \tag{82}$$

Now, when $d\theta/d\tau$ does not vanish, the first term on the left-hand side of equation 82 is insignificant with respect to the second term, N_p is a large number, and the above equation for larger times may be replaced by

$$-(1 + \theta) \frac{d\pi}{d\tau} + \frac{N_p}{r} \frac{d\theta}{d\tau} = \pi^{1/2} \tag{83}$$

63. In the example of a numerical analysis of the pneumatic generator action the above form of the equation will be used to represent the flow of air into the pneumatic chamber.

PART V: INITIAL CONDITIONS OF CHANGING PRESSURES

General Relations in Air and Water Flows

64. The discussions given in the preceding sections refer to the later stages of the pressure buildup in the pneumatic chamber, after the chamber has been opened to the air outside. For these later times the increase of pressure is uniform, and the accompanying fall of water surface in the chamber is also uniform. For the initial times, just after the chamber is opened to air outside, the pressure change within the chamber is undulatory. First, there is an increase of pressure of considerable magnitude, and next, a pressure fall. These are followed by cyclic changes of gradually decreasing magnitudes.

65. Consider once more the curve in fig. 5. Note that the wave does not emerge from the chamber immediately after it is opened to air, but about 0.3 or 0.4 sec later. During this period of time the elevation of water in the chamber remains the same; that is, during this time interval V_0 is maintained. As a consequence, there is a large increase of pressure owing to entering air, over what it would have been if the water surface had started to fall immediately after the chamber was opened. The large increase of pressure forces the water out of the chamber almost suddenly and at a rate that has the effect of producing a vacuum, a depression of pressure. Similar effects are reproduced in the sequence but with lesser severity. The explanation is that the inertia of water is the underlying cause of the initial oscillations of the air pressure.

66. In a mathematical treatment dealing with the initial condition it is necessary to determine the flow from the chamber into the channel. The expression for the flow may be obtained from equation 7a. In this, write $g(H - \delta)$ in the place of Ω_1 , $-\Delta p$ in place of P , gy in place of Ω_2 , and $\rho g(H_0 + h - y)$ in place of P_0 . Assume that v_1 is independent of ℓ and u_2 independent of y . Thus, the energy equation is

$$\frac{\rho}{2}\int_A \frac{\partial v^2}{\partial t}\, dx\, dy = v_1 \ell \left(\frac{\rho}{2} v_1^2 + \rho g H - \rho g \delta - \Delta p\right)$$

$$- u_2(H_0 + h)\left[\rho g(H_0 + h) + \frac{\rho}{2} u_2^2\right] \quad (84)$$

If the flow through the nozzle is resisted because of friction, then Δp is to be replaced by $\Delta p + \Delta p_L$ where Δp_L , a positive quantity, represents the additional pressure difference to overcome the resistance. Introducing Δp_L and remembering that $v_1 \ell = u_2(H_0 + h)$, the result may be written

$$\frac{\rho}{2}\int_A \frac{\partial v^2}{\partial t}\, dx\, dy = v_1 \ell \left[\frac{\rho}{2} v_1^2 - \frac{\rho}{2} u_2^2 + \rho g(H - H_0) - \rho g h - \rho g \delta - \Delta p_L - \Delta p\right] \quad (85)$$

One may put

$$\frac{\rho}{2}\int_A \frac{\partial v^2}{\partial t}\, dx\, dy = \frac{\rho}{2}\frac{\partial}{\partial t} v_1^2 \bar{M}\ell^2 = \rho v_1 \ell^2 \bar{M}\frac{\partial v_1}{\partial t} \quad (86)$$

where $\bar{M}$ is a dimensionless quantity, the value of which depends on the form of the pneumatic chamber and the nozzle The determination of it is discussed in the Addendum. In view of this latter relation, equation 85 may now be written as

$$\rho \bar{M}\ell \frac{dv_1}{dt} = \frac{\rho}{2} v_1^2 - \frac{\rho}{2} u_2^2 + \rho g(H - H_0) - \rho g h - \rho g \delta - \Delta p_L - \Delta p$$

Dividing by $\rho g(H - H_0)$ or by its equivalent Δp_0 and remembering that $v_1 = d\delta/dt$,

$$\frac{\bar{M}\ell}{g(H - H_0)}\frac{d^2\delta}{dt^2} = \left[1 - \frac{1}{2}\frac{u_2^2 - v_1^2}{g(H - H_0)} - \frac{h}{H - H_0} - \frac{\delta}{H - H_0}\right.$$

$$\left. - \frac{\Delta p_L}{\rho g(H - H_0)} - \frac{\Delta p}{\Delta p_0}\right] \quad (87)$$

A simple transformation will show that

$$\frac{u_2^2 - v_1^2}{g(H - H_0)} = \frac{H_0}{H - H_0}\left(\frac{h}{H_0}\right)^2\left[1 - \frac{1}{2}\frac{h}{H_0} - \left(\frac{H_0}{\ell}\right)^2\right]$$

or with a sufficient degree of approximation

$$\frac{u_2^2 - v_1^2}{g(H - H_0)} = \frac{H_0}{H - H_0}\left(\frac{h}{H_0}\right)^2$$

And also, as it is shown in the Addendum,

$$\frac{\Delta p_L}{\rho g(H - H_0)} = \frac{H_0}{H - H_0}\,\frac{\zeta}{2}\left(\frac{h}{H_0}\right)^2\left(\frac{H_0}{D}\right)^2$$

so that

$$\frac{1}{2}\,\frac{u_2^2 - v_1^2}{g(H - H_0)} + \frac{\Delta p_L}{\rho g(H - H_0)} = \frac{H_0}{H - H_0}\left[\frac{1}{2} + \frac{\zeta}{2}\left(\frac{H_0}{D}\right)^2\right]\left(\frac{h}{H_0}\right)^2$$

Substituting these in the energy equation the latter becomes

$$\frac{\bar{M}\ell}{g(H - H_0)}\,\frac{d^2\delta}{dt^2} = 1 - \frac{\delta}{H - H_0} - \frac{h}{H - H_0} - \frac{H_0}{H - H_0}\left[\frac{1}{2} + \frac{\zeta}{2}\left(\frac{H_0}{D}\right)^2\right]\left(\frac{h}{H_0}\right)^2 - \frac{\Delta p}{\Delta p_0} \qquad (88)$$

Introducing the dimensionless variables

$$\eta = h/H_0$$

$$\theta = \delta/\Delta$$

$$\pi = \Delta p/\Delta p_0$$

$$\tau = t/t_0$$

where t_0 is a characteristic time, see equation 81, the energy equation becomes

$$\frac{d^2\delta}{dt^2} = A_1(1 - A_2\theta - A_3\eta - A_4\eta^2 - \pi) \qquad (89)$$

where

$$A_1^{-1} = \frac{\bar{M}\ell/\Delta}{g(H - H_0)t_0^2}$$

$$A_2 = \frac{\Delta}{H - H_0}$$

$$A_3 = \frac{H_0}{H - H_0}$$

and

$$A_4 = \frac{H_0}{H - H_0}\left[\frac{1}{2} + \frac{\zeta}{2}\left(\frac{H_0}{D}\right)^2\right]$$

The relation between δ and h results from the continuity condition

$$v_1 \ell = u_2(H_0 + h)$$

That is,

$$\ell \frac{d\delta}{dt} = \sqrt{gH_0} \cdot H_C \cdot \frac{h}{H_0}\left(1 + \frac{3}{4}\frac{h}{H_0}\right)$$

In terms of dimensionless variables

$$\frac{d\theta}{d\tau} = \frac{\sqrt{gH_0} \cdot H_0 t_0}{\ell\Delta}\, \eta\left(1 + \frac{3}{4}\eta\right) \tag{90}$$

Inverting

$$\eta = A_5 \frac{d\theta}{d\tau} - A_6\left(\frac{d\theta}{d\tau}\right)^2 \tag{91}$$

where

$$A_5 = \frac{\ell\Delta}{\sqrt{gH_0} \cdot H_0 t_0}$$

and

$$A_6 = \frac{3}{4} A_5^2$$

67. Equations 88 and 89 describe the flow of water out of the chamber. To these one must now add the equation describing the flow of air into the chamber; that is,

$$-(1 + \theta)\frac{d\pi}{d\tau} + A_8 \frac{d\theta}{d\tau} = A_7 \pi^{1/2} \tag{92}$$

which, in conformity with equation 83, yields

$$A_8 = \frac{N_p}{r}$$

and

$$A_7 = 1$$

68. Our study of the flow of air into the pneumatic chamber was not sufficiently complete to formulate a theoretical basis for determining A_8 and A_7 a priori. In the numerical analysis to be shown subsequently, the values of A_8 and A_7 will be inferred from test records that enable one to give π and θ as functions of τ.

69. Equations 90, 91, and 92 are the basic relations describing π, θ, and η as functions of τ. The desired solution may be from the appropriate difference equations shown below, using the technique of computers.

Initial conditions:

$$\theta = 0 \ , \ \eta = 0 \ , \ \pi = 1 \ , \ \tau = 0 \tag{93}$$

Differences: $\tau = n\Delta\tau$

$$\left.\begin{aligned} \left(\frac{d\theta}{d\tau}\right)_{n+1} &= \left(\frac{d\theta}{d\tau}\right)_n + \left(\frac{d^2\theta}{d\tau^2}\right)_n \Delta\tau \\ \theta_{n+1} &= \theta_n + \left(\frac{d\theta}{d\tau}\right)_n \Delta\tau \\ \eta_{n+1} &= \eta_n + \left(\frac{d\eta}{d\tau}\right)_n \Delta\tau \end{aligned}\right\} \tag{94}$$

Relations at $\tau = n\tau$:

$$\left.\begin{aligned} \left(\frac{d^2\theta}{d\tau^2}\right)_n &= A_1(1 - A_2\theta_n - A_3\eta_n - A_4\eta_n^2 - \pi_n) \\ \eta_n &= A_5\left(\frac{d\theta}{d\tau}\right)_n - A_6\left(\frac{d\theta}{d\tau}\right)_n^2 \\ \left(\frac{d\pi}{d\tau}\right)_n &= -\left[A_7\pi_n^{1/2} - A_8\left(\frac{d\theta}{d\tau}\right)_n\right] \div (1 + \theta) \end{aligned}\right\} \tag{95}$$

An Example of Numerical Evaluation

70. To see whether the relations developed in the previous section are sufficient to describe the actions of a pneumatic generator, the case of Run A_{10-2} will be considered. The graphs of the fall of water surface, the increase of internal pressure, and the height of the issuing wave were shown in fig. 5. First, the characteristic time t_0 as defined by equation 81 needs to be established by noting the variation of pressure in the chamber with time prior to the emergence of the wave from the generator. The data are shown in fig. 12, and the straight line drawn implies that

$$\frac{1}{2}\,KCa = 1.65 \text{ per sec}$$

and hence

$$t_0 = 0.3 \text{ sec} \quad \text{and} \quad \tau = 3.3t$$

To complete the listing of the pertinent quantities of the run, the following ones are added:

$H_0 = 0.319$ ft	$p_0/\rho g = 34.1$ ft
$H = 1.419$ ft	$\Delta p_0/\rho g = 1.094$ ft
$H_T = 1.885$ ft	$\gamma = 1.408$
$\Delta = 0.466$ ft	$\theta = 66$ F
$\ell = 2$ ft	$\zeta = 1.66$
	$\bar{M} = 6.99$

The generator is of the elevated type, and the appropriate values of ζ and $\bar{M}$ are discussed in the Addendum. With the above, the constants appearing in equations 89 and 91 are:

$A_1 = 2.06$	$A_4 = 0.682$
$A_2 = 0.424$	$A_5 = 3.11$
$A_3 = 0.284$	$A_6 = 7.28$

71. Next must be indicated the values of the constants A_7 and A_8 in equation 92. The connections to outside air and to aspirator were made through a so-called three-way valve. The characteristics of the air passage are not known. However, the limiting values of A_7 and A_8 can be established on the basis of equation 92 by inserting in it the limiting values of $d\pi/d\tau$ and $d\theta/d\tau$. The data of the observations are shown in fig. 13 in terms of the dimensionless variables. It is seen that $d\theta/d\tau$ and $d\pi/d\tau$ are practically constant for large values of τ. For $\tau = 10$, one has

$$\theta = 0.656$$

$$\pi = 0.676$$

$$d\theta/d\tau = 0.0624$$

$$-d\pi/d\tau = 0.0254$$

Substituting these in equation 94 yields

$$0.822\ A_7 = 0.0421 + 0.0624\ A_8$$

a linear equation in the constants A_7 and A_8. If it is desired that equation 92 yield the pressure increases observed during the initial portion of the times, one must have

$$A_7 = 1$$

and hence

$$A_8 = 12.7$$

72. It would be instructive to inquire if there are differences between the values of A_8 for the moderate values of τ and the limiting value just given. Since $A_8 = \frac{N_p}{r}$, any variation in A_8 would indicate a corresponding variation in r: that is, the ratio of the effective coefficient C to its initial value C_0. Through equation 90, expressing $d\theta/d\tau$ in terms of η, one has from equation 92

$$A_8 = A_5 \frac{\pi^{1/2} + (1 + \theta)d\pi/d\tau}{\eta\left(1 + \frac{3}{4}\eta\right)} \tag{96}$$

The values of A_8 computed on this basis are given in fig. 13. The straight line drawn in the figure shows the limiting value of $A_8 = 12.7$. The value of A_8 for moderate τ oscillates about the limiting value. This fact will be ignored, and the numerical analysis will be attempted assuming that A_8 is a constant throughout.

73. A Fortran program for the numerical solution of equations 93, 94, and 95 is shown in table 7. The numerical results are plotted in fig. 14 and may be compared with the observational values. The initial and final values of π are correctly computed, except that for the moderate times there is a difference in the oscillation periods. This is due primarily to the circumstance that in reality A_8 is not a constant quantity as assumed for the computation. The final values of η and θ are also reproduced nearly correctly. There are, however, marked differences between the observed and computed values for the initial portion of time. This suggests that the relation between η and $d\theta/d\tau$ implied in equation 90 or equation 91 although valid for large values of τ is not sufficiently valid for the instant that the wave is emerging from the generator and accordingly for the rigor of the analysis the necessary modification needs to be introduced.

74. The above example of the analysis shows that in essence the mathematical formulation put forward is quite adequate to describe the mechanical action of pneumatic generators for surges. A better knowledge of the orifice or the air passage characteristics can lead to a more faithful representation of the actions transpiring. The analysis employed by Shen,[2] from the point of view of mechanics, is equivalent to the present analysis. There are, however, differences in the mathematical details arising from the fact that in establishing the flow equations Shen considered the particle velocity at the nozzle mouth whereas the present analysis considered the wave height.

PART VI: DESIGN OF PNEUMATIC TANKS FOR SINGLE SURGES

75. The preceding discussion of experiments and also of theory has shown that pneumatic tanks would serve well for the generation of elongated surges moving with constant heights and constant depths. The required controls and manipulations are truly simple. The channel waters are raised to prescribed heights in the chamber by suction, and the chamber air suction pressure is reduced by allowing the air from outside to enter the chamber through an aperture, preferably a circular orifice, the aperture opening remaining the same during the filling. In this process marked oscillation occurs both in the air pressure in the chamber and in the generated wave, due to inertia effect. This oscillation is of short duration, and soon steady conditions are established. The pressure increases uniformly in association with the wave generation of the required form. It is more simple to base the pneumatic tank design with reference to these later conditions. With this understanding the relations needed for the design are the following:

$$\frac{h}{H_0} = \left(\frac{H}{H_0} - 1\right)\left(1 - \frac{1}{4}\frac{h}{H_0}\right)\frac{k_1}{1 + \lambda} \qquad \text{(A)}$$

$$2k_2 = \frac{h}{H_0}\frac{\sqrt{gH_0}}{(H - H_0)}\left(1 + \frac{3}{4}\frac{h}{H_0}\right) \cdot \frac{H_0}{\ell} \qquad \text{(B)}$$

$$\frac{2k_2}{k_1} = \frac{1}{1 + \lambda}\frac{\sqrt{gH_0}}{\ell}\left(1 + \frac{1}{2}\frac{h}{H_0}\right) \qquad \text{(C)}$$

$$\frac{\Delta p}{\Delta p_c} = 1 - k_1 - 2k_2 t \; ; \; \frac{\Delta p_0 - \Delta p_i}{\Delta p_0} = k_1 \qquad \text{(D)}$$

$$\Delta p_0 = \rho g(H - H_0) \qquad \text{(E)}$$

$$2k_2 = \frac{KCa}{\left(1 + \gamma\frac{H - H_0}{H_T - H} \cdot \frac{p}{\Delta p_0}\right)} \qquad \text{(F)}$$

$$K = \frac{c_0 \gamma^{1/2}}{V_0} \left(\frac{p_0}{\Delta p_0} \right)^{1/2} \quad \underline{\text{and}} \quad C = 1 \tag{G}$$

$$u_2 = \sqrt{gH_0} \, \frac{h}{H_0} \left(1 - \frac{1}{4} \frac{h}{H_0} \right) \tag{H}$$

$$a = \sqrt{gH_0} \left(1 + \frac{3}{4} \frac{h}{H_0} \right) \tag{I}$$

$$\delta \cdot \ell = at \cdot h \tag{J}$$

76. Terms and symbols, although defined in the Glossary, are repeated here for easy reference. See fig. 15.

a	Area of orifice prorated per foot of chamber width, ft^2. A, total area of openings; B, section width. $a = A/B$
C	Coefficient of discharge for apertures
c_0	Velocity of sound for outside air, ft/sec, $= 49.1 \cdot \sqrt{\theta}$. θ . temperature in degrees Rankine $= 460 + {}^{\circ}F$
D	Depth of the narrowest part of the deflecting nozzle, ft
g	Constant of gravity, ft/sec^2
h	Wave height at the nozzle mouth, ft
H	Elevation of water in the pneumatic chamber at the start of the wave, ft
H_0	Undisturbed depth of water at nozzle mouth, ft
H_T	Elevation of pneumatic chamber ceiling, ft
ℓ	Length of chamber in the longitudinal direction, ft
p_0	Atmospheric pressure outside of chamber, $poundals/ft^2$
u_2	Wave particle velocity at nozzle mouth, ft/sec
V_0	Initial volume of air in chamber per foot of chamber length, ft^3. $V_0 = \ell(H_T - H) \cdot 1$
γ	Ratio of specific heats. For air $\gamma = 1.40$

h Fall of water surface in chamber, ft

Δp Suction pressure in chamber during the subsequent wave motion, poundals/ft^2

Δp_i Suction pressure in chamber at the beginning of wave motion, poundals/ft^2

Δp_0 Initial suction pressure in chamber to raise water to height H, poundals/ft^2 (Note: Δp_0 is a positive quantity)

λ Friction factor of nozzles

ρ Density of water, lb/ft^3

ω Wave velocity, ft/sec

77. Once the design wave height h is selected for a given depth of channel water H_0, equation A is the relation to determine the height H to which the water in the chamber must be raised. For the evaluation, k_1 is assumed, and the resistance factor λ is assigned a very likely value computed by the relation

$$\lambda = 0.83\left(\frac{H_0}{D}\right)^2 \cdot \frac{h}{H_0}$$

discussed in the Addendum. Here D is the depth of the narrowest part of the deflecting nozzle. A good selection for k_1 would be a value of 0.07 up to 0.10 or 0.12.

78. The significance of k_1 is found from equation D which gives the law of variation of pressure in the pneumatic chamber air space for the generation of an elongated wave of constant depth. The effective suction pressure at the start of wave motion is Δp_i, a quantity less than Δp_0, the suction pressure to raise the water to a height H. Equation E gives the suction pressure Δp_0 in terms of H and H_0. Now $\Delta p_i = (1 - k_1)\Delta p_0$. The multiplier $2k_2$ gives the rate of change of pressure in the pneumatic chamber air space and represents the quantity $\frac{d}{dt}\Delta p/\Delta p_0$, where Δp is the suction pressure during formation of the wave. Equation C gives the value of the ratio $2k_2/k_1$ in terms of the wave height and the length of the generator in the longitudinal direction.

79. Equation F shows that the value of $2k_2$ is dependent on the area of the apertures that allow air to enter the pneumatic chamber. Because of this relation one has a certain latitude in selecting the value of k_1. Unfortunately, relation F is not immediately applicable to pneumatic wave generators of a type where entrance of air into the chamber is through tortuous valves rather than through a circular orifice as in some of the laboratory tests. For such cases the form of equation F, although qualitatively correct, is not applicable quantitatively, since the appropriate value of C is not known. This means that each pneumatic chamber utilizing various forms of valves needs to be individually calibrated. This will cause no difficulty if in the design of the valves provisions are made to change the valve opening areas freely and arbitrarily.

80. Such factors as particle velocity, wave celerity, discharges, and discharge rates may be directly obtained from equations H, I, and J if desired.

81. When designing the pneumatic generator to be used in the Hilo Bay tsunami model the above procedure was used. A sketch of the generator as originally designed is shown in fig. 16. Assuming that the limiting wave heights in the central portion of the bay shall never exceed 50 ft, and that the depth of water at the bay mouth is about 300 ft, one has $h/H_0 = 0.167$. In the model $H_0 = 1.5$ ft, and therefore the design wave height is $h = 0.25$ ft. The base of the model is 80 ft, and the superficial area 1600 ft^2. The volume of the wave would be 5 ft^3, prorated per foot of length of the mouth.

82. The depth of the nozzle throat was chosen as 1 ft. Applying equation 143, $\lambda = 0.31$. It was noted during the experimental study that k_1 had a value close to 0.075 for the better runs made with the elevated generators. We shall accept this value. Placing $k_1 = 0.075$, $h/H_0 = 0.167$, and $\lambda = 0.31$, equation A yields $H = 6.0$ ft. This is the height the water can be raised in the chamber. As a reasonable value, H_B was put equal to 4.5. Since the maximum storage will be $(H - H_B)$ and since this equals 5 ft^3, the required chamber length would be $\ell = 6$ ft. The air gap is arbitrary and therefore was assigned the value of $\Delta = 2$ ft which makes $H_T = 8$ ft, the height of the generator.

The nozzle was 3.5 ft in length and was recessed in order to have a compact generator. A curved surface was introduced behind the nozzle to reduce the losses of deflection.

83. To know approximately what size circular aperture might be needed for the passage to outside air, resort may be made to equations F and G. First, one must determine $2k_2$. From equation B, $2k_2 = 0.071$; hence $2k_2/k_1 = 0.95$, which is in agreement with equation C. In view of the quantities $p_0/\Delta p_0 = 7.5$, $c_0 = 1120$ ft/sec, $V_0 = 12$ ft^3, equation G yields $K = 363$ per sec ft^2.* Returning to equation F, the required aperture area is 0.0048 ft^2, the value prorated per foot of chamber width. If the chamber width is 10 ft and only one aperture is used, the area of the aperture would be 0.045 ft^2.

84. It is discussed in the Addendum that the influence of the air jet from the orifice impinging upon the surface of water would be quite insignificant, if the air gap is about ten times the diameter of the circular aperture. For the above value the diameter is 0.24 ft; thus the air gap selected is satisfactory.

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* Another way of expressing this would be $K^{-1} = 0.00276$ sec ft^2.

ADDENDUM

This Addendum presents a few results of general nature, especially in regard to the treatment of flows (gas and liquid), which may prove to be of assistance in clarifying the numerous applications made in the main text.

1. Eulerian form of energy equation. The Eulerian form of energy equation (that is, the form applicable in an area within a fixed boundary not moving with the liquid) may be obtained from the Lagrangian form (that is, the form applicable to the same liquid portion in motion). Lamb[3] gives the derivation for the latter. However, there is some interest if the derivation is made in another manner. Consider the equations of motion for a two-dimensional field:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{97}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{98}$$

Together with these one also has the condition of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{99}$$

or

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{100}$$

Imagine that in the velocity field of fig. 2b one takes a closed curve s fixed in space and of area S. Since the field is two-dimensional, the flow is into and out of a cylindrical surface which is normal to the plane of the flow lines. The curve s is the intersection of the cylinder with the plane $z = 0$. Take the length of the cylinder to be unity. Multiply equation 97 by ρu, equation 98 by ρv, and add. Hence,

$$\frac{1}{2}\rho\frac{\partial}{\partial t}q^2 + \frac{1}{2}u\rho\frac{\partial q^2}{\partial x} + \frac{1}{2}v\rho\frac{\partial q^2}{\partial y} = -\rho u\frac{\partial \Omega}{\partial x} - \rho v\frac{\partial \Omega}{\partial y} - u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y}$$

where

$$q^2 = u^2 + v^2 \tag{101}$$

The last equation may be written also as

$$\frac{1}{2}\rho\frac{\partial}{\partial t}q^2 = -\frac{1}{2}\frac{\partial}{\partial x}(\rho uq^2) - \frac{1}{2}\frac{\partial}{\partial y}(\rho vq^2) + \frac{q^2}{2}\left(\frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y}\right)$$

$$- \frac{\partial}{\partial x}(\rho u\Omega) - \frac{\partial}{\partial y}(\rho v\Omega) + \Omega\left(\frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y}\right)$$

$$- \frac{\partial}{\partial x}(up) - \frac{\partial}{\partial y}(vp) + p\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)$$

which in view of the continuity relation, equation 100, simplifies to

$$\frac{1}{2}\rho\frac{\partial}{\partial t}q^2 + \frac{q^2}{2}\frac{\partial\rho}{\partial t} + \Omega\frac{\partial\rho}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial x}(\rho uq^2) - \frac{1}{2}\frac{\partial}{\partial y}(\rho vq^2)$$

$$- \frac{\partial}{\partial x}(\rho u\Omega) - \frac{\partial}{\partial y}(\rho v\Omega)$$

$$- \frac{\partial}{\partial x}(up) - \frac{\partial}{\partial y}(vp) + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Multiply the two sides of this equation by $dx\,dy$ and integrate over the surface S delineated by the curve s. Effect the integration using the Green's theorem:

$$\int_s(\ell U + mV)ds = -\int_S\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)dS$$

where ℓ and m are the directional cosines of the normal drawn inward to the curve s, and U and V each are functions of x and y. One has

$$\int_S\left(\frac{1}{2}\frac{\partial}{\partial t}\rho q^2 + \Omega\frac{\partial\rho}{\partial t}\right)dS = \int_s\left(\rho\frac{q^2}{2} + \Omega\rho + p\right)(\ell u + mv)ds$$

$$+ \int_S p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dS \tag{102}$$

If the liquid is incompressible, $\frac{\partial\rho}{\partial t} = 0$ and the above relation simplifies to

$$\int_S \frac{\rho}{2} \frac{\partial q^2}{\partial t} \, dS = \int_s \left(\frac{q^2}{2} + \Omega\rho \right) v_n ds + \int_s p v_n ds \tag{103}$$

where

$$v_n = \ell u + mv$$

This is the Eulerian form of energy equation for incompressible liquids. The interpretation is that the rate of change of kinetic energy in a bounded region is equal to the difference in the potential and kinetic energies of the liquids entering and leaving the boundary and the rate of work done by the pressure on the periphery.

2. <u>Internal energy of gases.</u> Denote the internal energy of gas per unit mass by ϵ. Thermodynamics suggests two processes by which the internal energy of a gas may be determined. In the first, the gas is made to expand without the inflow or outflow of heat. Thus,

$$p \, dv + d\epsilon = 0 \; ; \; v = \frac{1}{\rho}$$

Hence,

$$\epsilon = -\int p \, d\frac{1}{\rho} = \int p \frac{d\rho}{\rho^2} \tag{104}$$

and utilizing the adiabatic relation

$$p\rho^{\gamma} = \text{constant}$$

one has

$$\epsilon = \frac{1}{\gamma - 1} \frac{p}{\rho} + \epsilon_0 \tag{105}$$

In the second process heat is added while the volume is kept constant. Thus,

$$dQ = d\epsilon$$

Now,

$$dQ = C_v d\theta$$

where C_v is the specific heat at constant volume and $d\theta$ is the change in the absolute temperature. Accordingly

$$\epsilon = C_v\theta + \epsilon_0 \tag{106}$$

For a perfect gas $\theta = p/(R\rho)$, $C_p = R + C_v$, and $\gamma C_v = C_p$. And accordingly the expressions in equations 99 and 100 are equivalent to each other. Directly from equation 98

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} &= \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} \\ \frac{\partial \epsilon}{\partial x} &= \frac{p}{\rho^2}\frac{\partial \rho}{\partial x} \\ \frac{\partial \epsilon}{\partial y} &= \frac{p}{\rho^2}\frac{\partial \rho}{\partial y} \end{aligned} \tag{107}$$

3. <u>Energy equation for gases.</u> Lamb[3] shows that the energy equation for gases is

$$\frac{d}{dt}(T + V + W) = \int_s p(\ell u + mv)ds$$

where

$$W = \int_S \rho\epsilon \, dS \; , \; T = \int_S \frac{1}{2}\rho q^2 dS \; , \; V = \int_S \rho\Omega \, dS$$

This is the Lagrangian form. To obtain the Eulerian form consider the last integral in equation 102. Using the continuity condition, equation 100,

$$\int_S p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dS = -\int_S \left(\frac{p}{\rho}\frac{\partial \rho}{\partial t} + \frac{p}{\rho}u\frac{\partial \rho}{\partial x} + \frac{p}{\rho}v\frac{\partial \rho}{\partial y}\right)dS$$

and from equation 107

$$\int_S p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dS = -\int_S \left(\rho\frac{\partial \epsilon}{\partial t} + \rho u\frac{\partial \epsilon}{\partial x} + \rho v\frac{\partial \epsilon}{\partial y}\right)dS$$

or

$$\int_S p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dS = -\int_S \rho\frac{\partial \epsilon}{\partial t}\, dS + \int_S \epsilon\left(\frac{\partial u\rho}{\partial x} + \frac{\partial v\rho}{\partial y}\right) dS$$

$$-\int_S \left[\frac{\partial}{\partial x}(\rho u\epsilon) + \frac{\partial}{\partial y}(\rho v\epsilon)\right] dS$$

Using the continuity equation, equation 99, and the theorem of Green

$$\int_S p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dS = -\int_S \frac{\partial}{\partial t}(\rho\epsilon) dS + \int_s (\ell u + mv)\rho\epsilon\, ds$$

Introducing this result in equation 102, one has

$$\int_S \frac{\partial}{\partial t}(\rho q^2 + \rho\epsilon + \rho\Omega) dS = \int_s \left(\rho\frac{q^2}{2} + \Omega\rho + \epsilon\rho\right) v_n ds + \int_s p v_n ds \qquad (108)$$

In this $\rho\Omega$ is negligible in comparison with $\rho\epsilon$. Consider the expression

$$\Omega/\epsilon$$

which is the ratio of potential energy of gas at a point per unit mass to the internal energy of the same mass. As Ω may be measured from any horizontal level, let the plane of reference be the plane passing through the center of the vessel containing the gas and let y_m be the maximum displacement of a gas molecule from this plane. Accordingly, Ω is gy_m and

$$\frac{\Omega}{\epsilon} = \frac{gy_m}{\epsilon}$$

or

$$\frac{\Omega}{\epsilon} = (\gamma - 1)\frac{gy_m\rho}{p} = (\gamma - 1)\frac{gy_m}{R\theta}$$

or

$$\frac{\Omega}{\epsilon} = \gamma(\gamma - 1)\frac{gy_m}{c_0^2}$$

where c_0 is the velocity of sound for the gas. Since gy_m/c_0^2 is a very small quantity, it is obvious that Ω is negligible in comparison with ϵ . With this understanding equation 108 reduces to

$$\int_S \frac{\partial}{\partial t}(\rho q^2 + \rho\epsilon)dS = \int_s \left(\rho\frac{q^2}{2} + \epsilon\rho\right)v_n ds + \int_s pv_n ds \qquad (109)$$

which is the Eulerian form of energy equation for gases. The interpretation is that the change in the kinetic and internal energy of a gas in a fixed volume S.1 is equal to the work done by the pressures on the boundary and the difference in the inflow and the outflow of internal and kinetic energies through the boundary of area s.1.

It is shown in the above that the statement of energies is consistent with the equations of motion in the absence of viscous dissipation or of turbulence and also when there is no heat flow through the solid boundaries. Similar derivations may be repeated for the cases involving viscous dissipation or turbulence. However, for the problem at hand it is just as well that the statement of the energies is excepted a priori.

4. Energy method to determine the flow of air into a closed vessel. The previous derivation leading to equation 62 was based on the condition of continuity of mass assuming a definite relation between the density and the pressure of air in the vessel. One may examine the flow of air into the same vessel using the method of energy. This may provide new information on the relation between density and pressure.

Let E be the total energy of the air, kinetic and intrinsic, contained in a vessel of volume V_0 at time t. In the absence of flow of heat through the solid boundaries

$$\frac{\partial E}{\partial t} = E_1 + E_2 \qquad (110)$$

where E_1 is the rate at which work is being done on the opening by the pressures and E_2 is the rate of convection of total energy across the boundaries and in this instance through the opening. This is in accordance with the general result in equation 109. Let p_0 and ρ_0 be the pressure and density of air outside, p_1 and ρ_1 the pressure and density inside, u_1 the velocity at the entrance, and σ_1 the cross-sectional area of the vena contracta. Now,

$$E_1 = p_1 u_1 \sigma_1$$

and

$$E_2 = \left(\frac{1}{2} u_1^2 + \frac{1}{\gamma - 1} \frac{p_1}{\rho_1}\right) \rho_1 u_1 \sigma_1$$

Hence,

$$E_1 + E_2 = u_1 \sigma_1 \left(\frac{\gamma}{\gamma - 1} p_1 + \frac{1}{2} \rho_1 u_1^2\right)$$

or

$$E_1 + E_2 = u_1 \sigma_1 p_0 \left(\frac{\gamma}{\gamma - 1} \frac{p_1}{p_0} + \frac{1}{2} \rho_1 \frac{u_1^2}{p_0}\right) \qquad (111)$$

Assuming that the flow of air into the vessel is adiabatic

$$u_1^2 = \frac{2\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \left[1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma - 1}{\gamma}}\right]$$

and

$$\frac{\rho_1}{\rho_0} = \left(\frac{p_1}{p_0}\right)^{\frac{1}{\gamma}}$$

Put $p_1 = p_0 - \Delta p$ and assume that the square of $\Delta p / p_0$ is negligible in comparison with unity, with this approximation

$$\left.\begin{aligned} u_1^2 &= 2\Delta p / \rho_0 \\ \frac{\rho_1}{\rho_0} &= 1 - \frac{1}{\gamma} \frac{\Delta p}{p_0} \\ \frac{p_1}{p_0} &\approx 1 - \frac{\Delta p}{p_0} \end{aligned}\right\} \qquad (112)$$

Substituting in equation 111

$$E_1 + E_2 = u_1 \sigma_1 p_0 \left[\frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} \frac{\Delta p}{p_0} + \frac{1}{\gamma}\left(\frac{\Delta p}{p_0}\right)^2\right]$$

Since $(\Delta p / p_0)^2$ is negligible in comparison with unity, finally

$$E_1 + E_2 = u_1 \sigma_1 p_0 \left(\frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} \frac{\Delta p}{p_0} \right)$$

Introducing the approximate value of u_1 from equation 112

$$E_1 + E_2 = \sqrt{2}\, \sigma_1 p_0 \sqrt{\frac{\Delta p}{\rho_0}} \left(\frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} \frac{\Delta p}{p_0} \right)$$

Since

$$c_0^2 = \gamma p_0 / \rho_0$$

$$E_1 + E_2 = \sqrt{2}\, \sigma_1 \left(\frac{p_0}{\gamma} \right)^{\frac{1}{2}} \left(\frac{\gamma}{\gamma - 1} - \frac{1}{\gamma - 1} \frac{\Delta p}{p_0} \right) \sqrt{\Delta p} \qquad (113)$$

The jet of air entering into the vess[illegible]most immediately downstream of the opening undergoes a turbulent expansion. This circumstance introduces an uncertainty in the estimate of internal energy inside if the volume occupied by the expanding jet is comparable to the volume of the vessel. On the other hand, if the vessel volume is many times larger than the volume occupied by the expanding jet, the errors in the energy estimate will be small. The uncertainty comes about by one's inability to account properly for the kinetic energy in the expanding jet. This can be broken into two parts, the kinetic energy of the mean flow at a point and the kinetic energy of the turbulent fluctuations. As the latter energy goes into heat, one should be concerned with the kinetic energy of the mean flow. This will be neglected if the volume V_j of the jet is a small fraction of V_0 , the volume of the vessel. Thus $E = E_i$, where E_i denotes the intrinsic energy of the air contained in the vessel. Now

$$E_i = C_v \theta_1' m$$

and since

$$m = V_0 \rho_1'$$

$$E_i = V_0 C_v \theta_1' \rho_1'$$

or

$$E_i = \frac{V_0}{\gamma - 1} R\theta_1'\rho_1'$$

But

$$R\theta_1'\rho_1' = p' = p_0 - \Delta p$$

and thus

$$E_i = \frac{V_0}{\gamma - 1} (p_0 - \Delta p) \tag{114}$$

and differentiating with time

$$\frac{dE}{dt} = - \frac{V_0}{\gamma - 1} \frac{d\Delta p}{dt} \tag{115}$$

Substituting equations 113 and 115 in equation 110 and dividing the resulting equation by $V_0 \Delta p_0$, the result is

$$- \frac{d}{dt} \frac{\Delta p}{\Delta p_0} = \sqrt{2}\, \sigma_1 \frac{c_0 \gamma^{1/2}}{V_0} \left(\frac{p_0}{\Delta p_0}\right)^{1/2} \left(1 - \frac{1}{\gamma} \frac{\Delta p}{p_0}\right) \sqrt{\frac{\Delta p}{\Delta p_0}}$$

Writing

$$C = \sqrt{2} \frac{\sigma_1}{a}$$

$$K = \frac{c_0 \gamma^{1/2}}{V_0} \left(\frac{p_0}{\Delta p_0}\right)^{1/2}$$

and neglecting the small fraction in the last set of parentheses of the right-hand side

$$- \frac{d}{dt} \frac{\Delta p}{\Delta p_0} = KCa \sqrt{\frac{\Delta p}{\Delta p_0}} \tag{116}$$

and this is the same equation as the one previously derived, equation 62. The implication is that the density relation assumed in the first derivation

$$\frac{\rho_1}{\rho_0} = 1 - \frac{1}{\gamma}\frac{\Delta p}{p_0}$$

is in accordance with the principle of energy where the volume occupied by the jet, V_j , is very small in comparison with V_0 , the volume of vessel.

If the volume of the receiving vessel is small, account must be made for the internal kinetic energy. This may be done as follows. Let u' be the velocity of air at any point. The internal kinetic energy now is

$$E_K = \frac{1}{2}\int_0^{V_0} \rho_1'(u')^2\, dV_0$$

where dV_0 is a volume element. If u_{10} is the initial entrance velocity, one may express the air velocities inside in terms of this initial velocity. Thus,

$$E_K = \frac{1}{2} u_{10}^2 V_0 \rho_0 \int_0^1 \left(\frac{u'}{u_{10}}\right)^2 \frac{dV_0}{V_0}$$

ignoring the small differences between ρ_0 and ρ_1 . Since $u_{10}^2 = 2\Delta p_0/\rho_0$, a more useful expression of E_K is

$$E_K = \alpha \frac{V_0}{\gamma - 1}\Delta p_0 \qquad (117)$$

where

$$\alpha = (\gamma - 1)\int_0^1 \left(\frac{u'}{u_{10}}\right)^2 \frac{dV_0}{V_0}$$

Accordingly, for the condition the total internal energy is

$$E = \frac{V_0}{\gamma - 1}(p_0 - \Delta p + \alpha\Delta p_0) \qquad (118)$$

and, hence,

$$\frac{dE}{dt} = -\frac{V_0}{\gamma - 1}\left(\frac{d\Delta p}{dt} - \Delta p_0\frac{d\alpha}{dt}\right) \qquad (119)$$

Proceeding as before and omitting steps that are obvious, one now has

$$- \frac{d}{dt}\frac{\Delta p}{\Delta p_0} + \frac{d\alpha}{dt} = KCa\sqrt{\frac{\Delta p}{\Delta p_0}} \tag{120}$$

which shows the effect of internal kinetic energy on the pressure changes in the filling of a receiving vessel. The quantity α remaining unknown the solution of the equation cannot be worked out. Speculate for a moment that

$$\alpha = -\beta\Delta p/\Delta p_0 \tag{121}$$

where β is a positive constant. In this it is tacitly taken that α increases with time whereas Δp decreases with time. Inserting this in the pressure equation above, one finds

$$- \frac{d}{dt}\frac{\Delta p}{\Delta p_0} = K\frac{C}{1+\beta}a\sqrt{\frac{\Delta p}{\Delta p_0}} \tag{122}$$

which accords with the experimental results that the effective coefficient of discharge in the cases of the vessel of small volumes is less than the values obtained with larger volumes.

A somewhat different situation arises when the lower part of the closed vessel contains water instead of a rigid base. In the case where the surface of water is close to the air opening at the top, the entering air in the form of a jet would be impinging on the water and then would be deflected. In this encounter work is done against water primarily by viscous tractions and also possibly by normal tractions. Let the energy consumed in this manner be of the rate E_v. In our inability to appraise the loss numerically, let it be supposed that it is proportional to the work done by the entering jet and the energy entering in; that is,

$$E_v = \alpha_2 (E_1 + E_2) \tag{123}$$

Accordingly, the energy balance equation is

$$\frac{\partial E}{\partial t} = (1 - \alpha_2)(E_1 + E_2) \tag{124}$$

in place of the one shown by equation 110. Repeating the analysis and omitting the steps of the transformations, one has

$$- \frac{d}{dt} \frac{\Delta p}{\Delta p_0} + \frac{d\alpha}{dt} = KCa(1 - \alpha_2) \sqrt{\frac{\Delta p}{\Delta p_0}} \tag{125}$$

and adding that $\alpha = -\beta \Delta p/\Delta p_0$, finally

$$- \frac{d}{dt} \frac{\Delta p}{\Delta p_0} = KCa \frac{1 - \alpha_2}{1 + \beta} \sqrt{\frac{\Delta p}{\Delta p_0}} \tag{126}$$

Thus, in these adopted views the effective coefficient of discharge depends on the volume of the vessel and the part of the surface of the water in the vessel in contact with the air jet.

5. Energy method to determine the flow of air into a vessel of changing volumes. Let V_0 be the volume at time t_0 and V the volume at time t . $V = V_0 + \Delta V$. Placing $\Delta V = \sigma_2 \delta$, where σ_2 is the area of the lower moving surface, the velocity of the surface is $v = d\delta/dt$. In agreement with equation 109 the energy relation is

$$\frac{\partial E}{\partial t} = (E_1 + E_2)_1 - (E_1 + E_2)_2 \tag{127}$$

where the second term on the right-hand side represents the rate of work done by the gas on the moving surface and the flow of energy, intrinsic and kinetic, associated with this motion. The terms $E_1 + E_2$ and $\frac{\partial E}{\partial t}$ have the same meaning and the same values as in the previous section except that V_0 will be replaced by V .

The work done by the air in the vessel on the moving surface is

$$E_{12} = p_1 \frac{d\delta}{dt} \sigma_2$$

and the outward flux of energies, kinetic and intrinsic, is

$$E_{22} = \left[\frac{1}{2}\left(\frac{d\delta}{dt}\right)^2 + \frac{1}{\gamma - 1} \frac{p_1}{\rho_1'}\right] \rho_1' \frac{d\delta}{dt} \sigma_2$$

Since the kinetic part is small in comparison with the intrinsic,

$$(E_1 + E_2)_2 = E_{12} + E_{22} = \frac{\gamma}{\gamma - 1} p_1 \frac{dV}{dt}$$

Equation 127 now may be written as

$$-\frac{V}{\gamma - 1} \frac{d\Delta p}{dt} + \frac{\gamma}{\gamma - 1} p_1 \frac{dV}{dt} = Cac_0 \left(\frac{p_0}{\gamma}\right)^{\frac{1}{2}} \frac{\gamma}{\gamma - 1} \sqrt{\Delta p}$$

Dividing by $V_0 \Delta p_0$, and writing $V = \left(1 + \frac{\delta}{\Delta}\right) V_0$, one has

$$-\left(1 + \frac{\delta}{\Delta}\right) \frac{d}{dt} \frac{\Delta p}{\Delta p_0} + \frac{N_p}{\Delta} \frac{d\delta}{dt} = KCa \sqrt{\frac{\Delta p}{\Delta p_0}} \qquad (128)$$

where

$$N_p = \gamma p_0 / \Delta p_0$$

and

$$K = \gamma^{1/2} \frac{c_0}{V_0} \left(\frac{p_0}{\Delta p_0}\right)^{1/2}$$

This is the relation for the rate of increase of pressure in a vessel of changing volumes. It is of the same form as equation 74, which was derived by assuming that the pressure and density relation is

$$\frac{\rho_1}{\rho_0} = 1 - \frac{1}{\gamma} \frac{\Delta p}{p_0}$$

One may modify this result according to the conditions that the volume of the receiving vessel is small and the moving surface is in the form of a water surface falling down. Introducing the corrections to take account of the internal kinetic energy E_K and the rate of dissipation from the surface contact E_v and repeating the steps of analysis as was done in the case of a closed vessel, and omitting details, one obtains

$$-\left(1 + \frac{\delta}{\Delta}\right)\left(\frac{d}{dt} \frac{\Delta p}{\Delta p_0} - \frac{d\alpha}{dt}\right) + \frac{N_p}{\Delta} \frac{d\delta}{dt} = KCa(1 - \alpha_2) \sqrt{\frac{\Delta p}{\Delta p_0}}$$

Assuming as before that $\alpha = -\beta \Delta p / \Delta p_0$

$$-\left(1+\frac{\delta}{\Delta}\right)\frac{d}{dt}\frac{\Delta p}{\Delta p_0}+\frac{1}{1+\beta}\frac{N_p}{\Delta}\frac{d\delta}{dt}=KCa\,\frac{1-\alpha_2}{1+\beta}\sqrt{\frac{\Delta p}{\Delta p_0}}$$

In view of the fact that α_2 and β are not known, it would be desirable to adopt the relation

$$-\left(1+\frac{\delta}{\Delta}\right)\frac{d}{dt}\frac{\Delta p}{\Delta p_0}+\frac{N_p}{\Delta}\frac{d\delta}{dt}=KCa\sqrt{\frac{\Delta p}{\Delta p_0}} \tag{128 bis}$$

to describe the variation of pressure in the receiving vessel with its volume changing with time, with the understanding that C, the effective discharge coefficient, could be a function of time.

6. Turbulent expansion of air jets. It was assumed in the development of the two preceding sections, that if the volume of the expanding jet is small in comparison with the volume of the vessel receiving the jet, then the internal energy of the air in the vessel is essentially intrinsic. What the jet size is, longitudinally and laterally, will now be discussed.

It is known that in a one-dimensional jet issuing from a slit, or in an axial jet issuing from a circular orifice, the width increases linearly with distance from the opening. The result from Förthman[9] on the width 2b of the expanding jet, the jet issuing from a slit, is given in fig. 17. Axial distance x is measured from a point upstream of the entrance and at a distance three times the width of the slit. Roughly, now,

$$b = 0.158x \tag{129}$$

The distributions of the velocities in the various normal sections are similar to each other. If u is the longitudinal component of the velocities at a point of distance y from the median plane, and u_m the velocity at the axis itself, then

$$\frac{u}{u_m}=f(y/b)$$

At the edges of the jet u vanishes and also the derivative du/dy. Subject to these conditions, one may adopt the expression

$$\frac{u}{u_m} = \frac{1}{2}\left(1 + \cos\frac{\pi}{b}y\right) \tag{130}$$

as a first approximation of the actual velocities. Since the variation of b with distance x is known, one may utilize the velocity expression to determine the value of u_m on the basis of momentum. It will be assumed that the results obtained for a one-dimensional jet apply also to an axial jet. Since the pressure is constant everywhere, the momentum of the liquid traversing a normal cross section should equal the momentum M_0 of the liquid entering the orifice. Accordingly,

$$\frac{M_0}{\rho_1} = 2\pi\int_0^b u^2 y\, dy$$

Introducing the value of the velocities from equation 130

$$\frac{M_0}{\rho_1} = \frac{2u_m^2 b^2}{\pi}\left(\frac{3}{2}\pi^2 - 4\right)$$

Since the area of the vena contracta is half the area of the orifice and u_1 is the entrance velocity,

$$\frac{M_0}{\rho_1} = u_1^2\sigma = \frac{\pi}{8}u_1^2 d$$

where d is the diameter of the orifice. Equating to each other

$$u_1^2 d^2 = \frac{32}{\pi^2}\left(\frac{3}{2}\pi^2 - 4\right)u_m^2 b^2$$

and in view of equation 129

$$\frac{u_m^2}{u_1^2} = 1.13\left(\frac{d}{x}\right)^2$$

In the pneumatic chamber, let the distance of the water surface from the orifice be x_s. The jet is deflected laterally after reaching the water surface. The pressure of impact at the axial point is denoted by

$$\Delta p_m = \rho_1\frac{u_m^2}{2}$$

and, thus,

$$\Delta p_m = 1.13 \frac{d^2}{x_s^2} u_1^2 \rho_1$$

Introducing u_1 from equation 112, the difference $\rho_0 - \rho_1$ is small, and

$$\Delta p_m/\Delta p = 1.13 d^2/x^2 \tag{131}$$

where Δp is the air pressure prevailing outside the jet and this is also the pressure in the vena contracta for a pneumatic chamber of large dimensions. The distance below the orifice such that Δp_m will be equal to $0.01\Delta p$ may be taken as the effective length X_j of the jet. Thus, the length of the jet on this basis is

$$X_j = 10.6d \tag{132}$$

And the volume of the jet,

$$V_j = 31.2d^3 \tag{133}$$

If d is 1 in., for example, the jet length X_j is 10.6 in., and the volume of jet V_j is 0.018 ft^3. If, on the other hand, Δp_m is chosen to equal 0.0029 Δp , then

$$X_j = 21.2d$$

and

$$V_j = 250d^3$$

The above results are for an isothermally flowing air. If $\Delta p/p_0$ is small the results may be applied also to a case of adiabatically moving air as in the pneumatic chamber since the errors in the application can be ignored.

7. Kinetic energy of liquid in chamber and nozzle. Previously in discussing the kinetic energy T_{21} in the pneumatic chamber and the

nozzle below, the energy was expressed as

$$T_{21} = \frac{\rho}{2} v_1^2 (H - H_B - \delta)\ell + \frac{\rho}{2} u_2^2 H_0^2 N_t \qquad (26 \text{ bis})$$

in which the first term on the right-hand side represents the kinetic energy in the upper part of the chamber, prismatic in shape, and therefore of constant cross section, and the second term represents the kinetic energy in the lower part of the chamber and the nozzle extension of it. Let the kinetic energy of the lower part be denoted by T_N ; that is,

$$T_N = \frac{\rho}{2} u_2^2 H_0^2 N_t \qquad (134)$$

where N_t is a numerical constant, its value being dependent on the shape of the nozzle and the type of the pneumatic generator, either low or elevated. For these two types the evaluation of N_t is carried out differently.

Consider first a low generator as in fig. 18a. The areas for which the kinetic energy T_N is to be evaluated consist of the area (1) and the area (2). Let q denote the velocity at a point in these areas, $q^2 = u^2 + v^2$. It suffices to write for a point in area (1)

$$v = v_1 y/D \quad \text{and} \quad u = v_1 x/D$$

and for a point in area (2)

$$u = q_0 H_0/b \quad \text{and} \quad b = D + (H_0 - D)x/\ell_t$$

where D is the depth of the nozzle throat, H_0 the depth of the nozzle mouth, ℓ_t is the length of the nozzle and q_0 the velocity at the nozzle mouth. Now,

$$T_N = \frac{\rho}{2} \int_A q^2 dA$$

where dA is an elementary area and A is the combined area of (1) and

(2). Introducing the above values of the velocity components, integrating, and in the result making use of the continuity condition

$$q_0 H_0 = v_1 \ell$$

one obtains

$$T_N = \frac{\rho}{2}\left[\frac{\ell_t}{H_0 - D}\log\frac{H_0}{D} + \frac{1}{3}\left(\frac{D}{\ell} + \frac{\ell}{D}\right)\right] v_1^2 \ell$$

Since $v_1 \ell = u_2(H_0 + h)$, also

$$T_N = \frac{\rho}{2}\left[\frac{\ell_t}{H_0 - D}\log\frac{H_0}{D} + \frac{1}{3}\left(\frac{D}{\ell} + \frac{\ell}{D}\right)\right]\left(1 + \frac{h}{H_0}\right)^2 u_2^2 H_0 \qquad (135)$$

Comparing with equation 135, then, for a low pneumatic chamber

$$N_t = \left(1 + \frac{h}{H}\right)^2 I_t \qquad (136)$$

where

$$I_t = \left[\frac{\ell_t}{H_0 - D}\log\frac{H_0}{D} + \frac{1}{3}\left(\frac{D}{\ell} + \frac{\ell}{D}\right)\right]$$

For an elevated pneumatic chamber the evaluation of N_t is done best by the method of graphics. Consider fig. 18b. Let s be the median curve of the deflector and b the section width. Obviously

$$T_N = \frac{\rho}{2}\int_0^s q^2 b \, ds$$

Since $qb = q_0 H_0$,

$$T_N = \frac{\rho}{2} q_0^2 H_0^2 \int_0^s \frac{ds}{b}$$

In view of $q_0 H_0 = u_2(H_0 + h)$, one has finally

$$T_N = \frac{\rho}{2} u_2^2 H_0^2 \left(1 + \frac{h}{H_0}\right)^2 \int_0^s \frac{ds}{b} \qquad (137)$$

Comparing this with equation 135, then, for an elevated pneumatic chamber

$$N_t = \left(1 + \frac{h}{H_0}\right)^2 I_t \tag{138}$$

where

$$I_t = \int_0^s \frac{ds}{b}$$

For the elevated pneumatic chamber used in the tests it is found that

$$I_t = 6.72$$

In the treatment of the initial condition of changing pressures it was convenient to introduce a numerical factor $\overline{M}$ in equation 86 of the meaning that

$$\frac{\rho}{2} v_1^2 \overline{M} \ell^2 = T_{21} = \frac{\rho}{2} v_1^2 (H - H_B - \delta)\ell + \frac{\rho}{2} u_2^2 H_0^2 N_t$$

Since $u_2(H_0 + h) = v_1 \ell$, the last relation can be written also as

$$v_1^2 \overline{M} \ell^2 = v_1^2 \ell^2 \left(\frac{H - H_B}{\ell} - \frac{\delta}{\ell} + I_t\right)$$

Hence,

$$\overline{M} = \frac{H - H_B}{\ell} - \frac{\delta}{\ell} + I_t$$

and for brevity

$$\overline{M} = \overline{M}_0 \left(1 - m \frac{\delta}{\Delta}\right) \tag{139}$$

where

$$\overline{M}_0 = \frac{H - H_B}{\ell} + I_t$$

and

$$m = \Delta / \ell M_0$$

During the emergence of the wave from the generator the numerical

factor $\bar{M}$ decreases with δ (that is, with the fall of the water surface in the chamber). For the elevated chambers the term containing δ may be ignored. To illustrate this we now evaluate $\bar{M}$ for the pneumatic chamber used in the tests and for the specific run A_{10-2} that was selected for the numerical analysis previously discussed. The pertinent data as seen in table 1 are:

$$H_0 = 0.319 \text{ ft} \qquad H_B = 0.872 \text{ ft}$$

$$H = 1.419 \text{ ft} \qquad H_T = 1.885 \text{ ft}$$

$$\ell = 2 \text{ ft} \qquad \Delta = 0.466 \text{ ft}$$

Equation 139 yields

$$\bar{M} = 6.99(1 - 0.032\ \delta/\Delta) \tag{140}$$

which shows that even when the fall of the water surface is twice the initial air gap the error in neglecting the δ/Δ term is only 6 percent.

8. Nozzle coefficient of friction. In dealing with the problem of chamber pressures, it became necessary to introduce in the energy equation a term to represent the total loss of energy ΔL during the time a wave having the length L and the height h is produced. As a matter of convenience, the loss was expressed in the form

$$\Delta L = \lambda \rho g h^2 N L \tag{52 bis}$$

that is, as a fractional part of the energy of the wave produced. The proportionality factor λ was referred to as the nozzle friction factor. If the losses are from viscosity, λ would be a constant independent of the wave height h. However, as it is readily understood, the losses originate from turbulence and for this reason λ would be dependent also on wave height. This is a matter that requires some elucidation.

From above, since $L = \omega t$, the rate of loss to be associated with the pneumatic chamber nozzle area is of the form

$$\frac{\Delta L}{\Delta t} = \lambda \rho g h^2 N \omega \tag{52a}$$

Since the loss in the region arises from two sources, the deflection and the subsequent expansion, the rate of loss may be expressed also as

$$\frac{\Delta L}{\Delta t} = \Delta P_L u_m D$$

where u_m is velocity in the narrow part of the throat, D the depth of the throat, and ΔP_L the differential energy head associated with the combined losses. It is customary to write

$$\Delta P_L = \zeta \frac{\rho}{2} u_m^2$$

where ζ is the loss coefficient. In this relation ζ is constant, its value depending on the form of the passages. Hence,

$$\frac{\Delta L}{\Delta t} = \zeta \frac{\rho}{2} u_m^3 D \tag{141}$$

From the condition of continuity $u_m D = u_2 H_0$,

$$\frac{\Delta}{\Delta t} = \zeta \frac{\rho}{2}\left(\frac{H_0}{D}\right)^2 u_2^3 H_0$$

Comparing this with equation 52a,

$$\lambda \rho g h^2 \omega = \zeta \frac{\rho}{2}\left(\frac{H_0}{D}\right)^2 u_2^3 H_0 \tag{142}$$

Ignoring secondary quantities,

$$H_0 u_2 = h\omega$$

and

$$u^2/gh = h/H_0$$

and hence equation 142 reduces to

$$\lambda = \frac{\zeta}{2}\left(\frac{H_0}{D}\right)^2 h/H_0 \tag{143}$$

which connects the loss coefficient ζ with the resistance factor λ .

Table 8 presents the values of λ for different runs with the elevated tank together with the wave height of the run and the water depth. The individual values for every run are the averages from two ways of evaluating λ . See tables 1 and 5. In the last column of table 8 are given the values of $\zeta/2$ reduced from λ using equation 143. The average value for ζ is

$$\zeta = 1.66$$

which is quite reasonable. Since the nozzle of the generator used is of low flare, the expansion losses would be small and the substantial part of the loss should be to the flow deflection through the throat bend. In this respect the bend action is similar to that in an elbow flow and for which the usually indicated coefficient is $\zeta = 1.13$.

An alternate expression for ΔP_L in terms of u_2 is

$$\Delta P_L = \zeta \frac{\rho}{2} \left(\frac{H_0}{D}\right)^2 \left(1 + 2\frac{h}{H_0}\right) u_2^2$$

which may be put in the form

$$\frac{\Delta P_L}{\rho g(H - H_0)} = \frac{H_0}{H - H_0} \cdot \frac{\zeta}{2} \left(\frac{h}{H_0}\right)^2 \cdot \left(\frac{H_0}{D}\right)^2 \qquad (144)$$

a result which was used in the section dealing with the general relations governing the flow of air into the pneumatic chamber and the flow of water out of the chamber.

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Table 1

Analysis of Elevated Pneumatic Generator Data

(Chamber connection to air through three-way valve)

Chamber width, 1 ft; length, 2 ft; flume width, 0.995 ft;

$p_0/\rho g$ = 34 ft

	Run A_{10-1}	Run A_{10-2}	Run A_{10-3}	Run A_{10-5}	Run A_{11-1}	Run A_{11-2}	Run A_{11-3}	Run A_{11-5}
	Observed							
h	0.052	0.051	0.048	0.042	0.051	0.050	0.052	0.053
H_0	0.308	0.319	0.338	0.364	0.303	0.303	0.302	0.305
H	1.500	1.419	1.317	1.119	1.470	1.470	1.469	1.490
$\Delta p_0/\rho g$	1.193	1.094	0.976	0.754	1.167	1.167	1.165	1.167
$\Delta p_i/\rho g$	1.109	1.021	0.917	0.704	1.085	1.094	1.091	1.089
$d/dt \Delta p_0/\rho g$	0.102	0.093	0.092	0.079	0.106	0.103	0.104	0.108
$d\delta/dt$	0.101	0.097	0.094	0.080	0.104	0.104	0.106	0.108
k_1	0.071	0.067	0.061	0.066	0.070	0.063	0.063	0.067
$2k_2$	0.085	0.084	0.094	0.111	0.093	0.088	0.090	0.093
u_2	0.547	0.512	0.477	0.390	0.571	0.570	0.583	0.586
H_T	1.885	1.885	1.885	1.885	1.885	1.885	1.885	1.885
	Computed							
$2k_2$	0.074	0.082	0.090	0.091	0.076	0.073	0.073	0.079
u_2	0.508	0.490	0.445	0.382	0.500	0.492	0.512	0.518
λ	0.43	0.40	0.14	0.08	0.31	0.22	0.18	0.24
λ	0.56	0.54	0.18	0.30	0.54	0.41	0.37	0.43

Table 2

Determination of Discharge Coefficient C

$V_0 = 4.03\ \text{ft}^3;\ \frac{p_0}{\rho g} = 34.1\ \text{ft},\ \theta = 66\ \text{F},\ c_0 = 1125\ \text{ft/sec}$

		C				
d, in.	t/d	$\frac{\Delta p_0}{\rho g} = 2.9$ ft	$\frac{\Delta p_0}{\rho g} = 1$ ft	Mean	$V_j/V_0 \times 10^3$	X_j/Δ
1.467	0.33	0.670	0.690	0.680	109.5	1.32
1.002	0.43	0.700	0.711	0.706	36.1	0.90
0.706	0.68	0.722	0.741	0.732	14.9	0.62
0.608	0.79	0.701	0.739	0.720	8.0	0.54
0.501	0.97	0.714	0.750	0.732	4.5	0.44
0.437	1.11	0.691	0.711	0.701	3.0	0.41
0.360	1.34	0.792	--	0.792	1.7	0.33
0.257	1.88	0.768	0.833	0.801	0.6	0.23

Table 3

Decrease of Discharge Coefficient with Volume

$\frac{p_0}{\rho g} = 34.1\ \text{ft},\quad \frac{\Delta p_0}{\rho g} = 1\ \text{ft}$

	d = 1.002 in.			d = 0.360 in.		
V_0 ft^3	C	X_j/Δ	$V_j/V_0 \times 10^3$	C	X_j/Δ	$V_j/V_0 \times 10^3$
4.03	0.720	0.90	36.1	0.820	0.32	1.66
3.05	0.672	1.19	47.8	0.743	0.43	2.20
2.00	0.592	1.81	73.2	0.696	0.66	3.35
1.46	0.551	2.49	10.0	0.564	0.90	4.59
0.92	0.383	4.09	15.9	0.482	1.48	7.00

Table 4

Determination of Discharge Coefficient from the Wave Data; Pneumatic Generator Tests

	Low Tank		Elevated Tank	
	Run A_{14-7}	Run A_{14-5}	Run A_{16-2}	Run A_{16-3}
H_0(ft)	0.297	0.291	0.298	0.300
H(ft)	1.070	1.076	1.471	1.469
H_T(ft)	1.609	1.609	1.885	1.885
ℓ(ft)	4.75	4.75	2.00	2.00
V_0(ft^3)	2.58	2.53	0.825	0.832
a(ft^2)	2.73×10^{-3}	1.36×10^{-3}	1.36×10^{-3}	2.81×10^{-3}
$\Delta p_0/\rho g$(ft)	0.771	0.775	1.171	1.168
K	3430	3480	8670	8600
KCa	3.06	1.80	6.1	13.5
Ka	9.28	4.70	11.80	24.20
C	0.332	0.383	0.516	0.556
d(in.)	0.706	0.500	0.500	0.717
Δ(in.)	2.5	2.5	5	5
X_j/Δ	5.96	4.22	2.12	3.04
$V_j/V_0 \times 10^3$	23.4	7.1	21.9	76.8

Table 5

Analysis of Pneumatic Generator Data

(Chamber connection to air through circular orifice)

$p_0/\rho g$ = 34 ft; θ = 66 F

	Low Chamber		Elevated Chamber	
	Run A_{14-7}	Run A_{14-5}	Run A_{16-2}	Run A_{16-3}
		Observed		
h	0.097	0.064	0.066	0.120
H_0	0.297	0.291	0.298	0.300
H	1.070	1.076	1.471	1.469
H_T	1.609	1.609	1.885	1.885
ℓ	4.75	4.75	2.00	2.00
V_0	2.58	2.58	0.825	0.825
a	2.73×10^{-3}	1.36×10^{-3}	1.36×10^{-3}	2.81×10^{-3}
$\Delta p_0/\rho g$	0.771	0.775	1.171	1.168
$\Delta p_i/\rho g$	0.680	0.724	1.081	0.875
$\frac{d}{dt}\Delta p/\rho g$	0.068	0.040	0.112	0.208
$d\delta/dt$	0.075	0.045	0.110	0.209
k_1	0.170	0.090	0.077	0.184
$2k_2$	0.089	0.052	0.095	0.217
K	3.43×10^3	3.48×10^3	8.67×10^3	8.60×10^3
		Computed		
λ	0.42	0.46	0.46	0.49
λ	0.25	0.08	0.29	0.62
C	0.85	1.01	0.92	1.01

Table 6

Effective Coefficient of Discharge of the

Experimental Pneumatic Chambers

	C			
	Low Chamber		Elevated Chamber	
t, sec	Run A_{14}-7	Run A_{14}-5	Run A_{16}-2	Run A_{16}-3
0	0.85	1.01	0.92	1.01
1	1.06	1.19	1.03	1.12
2	1.14	1.27	1.09	1.39
3	1.25	1.39	1.17	1.91
0*	0.33	0.38	0.52	0.56

* Determined for the initial instant of filling prior to the issuance of wave from chamber. The table indicates the discontinuity in value of the effective discharge coefficient in the action of the chamber.

Table 7

Fortran Program for Pneumatic Chamber Analysis

```
      PRINT 52
52    FORMAT (1H1,10X,4HTIME,10X,5HTHETA,13X,2HPI,12X,3HETA)
      TAU=0.0
      THE=0.0
      DTHE=0.0
      ETA=0.0
      PI=1.0
      READ 50,DELTAU,FINTAU,A1,A2,A3,A4,A5,A6,A7,A8
50    FORMAT (10F7.3)
3     PRINT 51, TAU,THE,PI,ETA
51    FORMAT (4F15.4)
      IF (TAU-FINTAU) 2,4,4
4     READ 50,DELTAU,FINTAU,A1,A2,A3,A4,A5,A6,A7,A8
      IF (DELTAU) 99,99,2
2     ETA=DTHE*(A5-A6*DTHE)
      D2THE=A1*(1.0-A2*THE-A3*ETA-A4*ETA*ETA-PI)
      DPI=-(A7*SQRTF(PI)-A8*DTHE)/(1.0+THE)
      THE=THE+DTHE*DELTAU
      PI=PI+DPI*DELTAU
      DTHE=DTHE+D2THE*DELTAU
      TAU=TAU+DELTAU
      GO TO 3
99    STOP
      END
```

Table 8

Resistance Coefficient ζ for Elevated Chamber Nozzle Flow

$D = 0.209$ ft

Run	h, ft	H_0, ft	λ	$\zeta/2$
A_{10-1}	0.052	0.308	0.45	1.10
A_{10-2}	0.051	0.319	0.47	1.22
A_{10-3}	0.048	0.338	0.16	0.41
A_{10-5}	0.042	0.364	0.19	0.50
A_{11-1}	0.051	0.303	0.42	1.14
A_{11-2}	0.050	0.303	0.31	0.86
A_{11-3}	0.052	0.302	0.28	0.75
A_{11-5}	0.053	0.305	0.34	0.92
A_{16-2}	0.066	0.298	0.38	0.81
A_{16-3}	0.120	0.300	0.51	0.59
			Mean	0.83

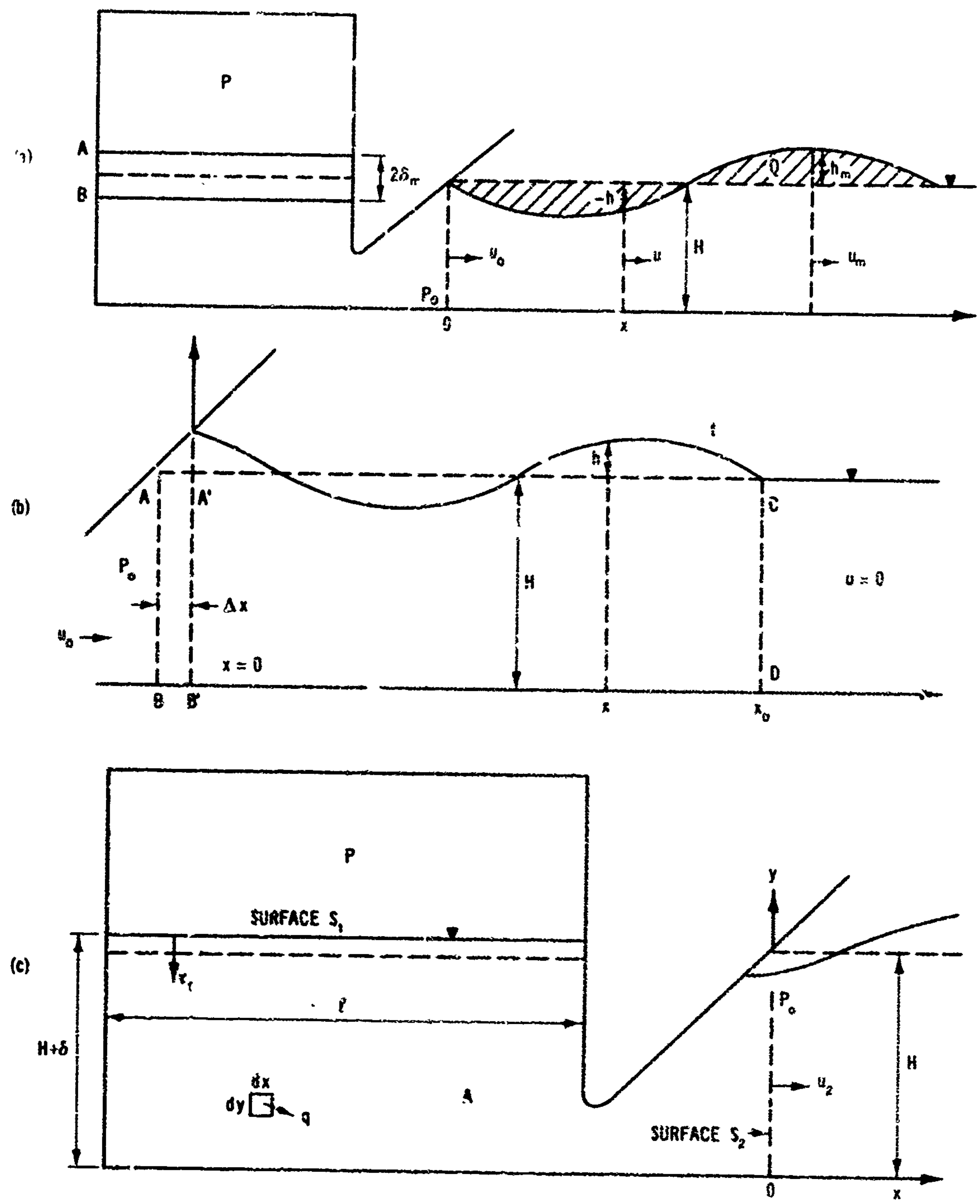
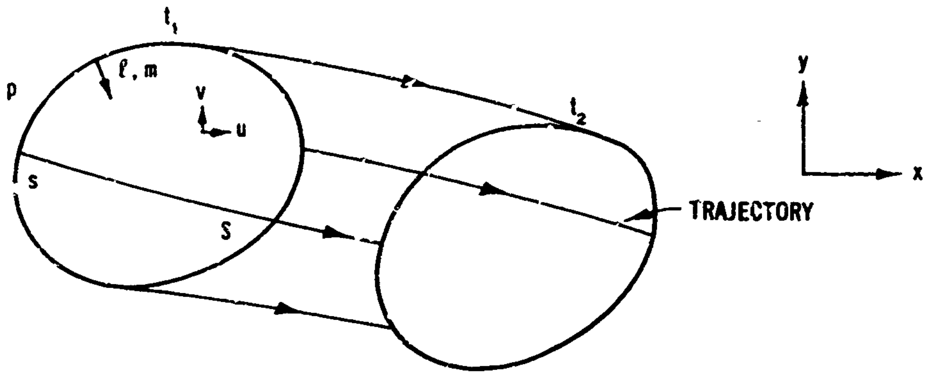


Fig. 1. Notation diagrams. Pneumatic generator; oscillatory shallow-water waves

(a) LAGRANGIAN VIEW:

$$(T+V)_2 - (T+V)_1 = \int_{t_1}^{t_2}\int_{s} (\ell u + mv)\, p\, ds\, dt$$

$$T = \frac{1}{2}\rho\int_{S} (u^2 + v^2)\, dS \qquad V = \frac{1}{2}\rho\int_{S} \Omega\, dS$$



(b) EULERIAN VIEW:

$$\frac{\rho}{2}\int_{S} \frac{\partial q^2}{\partial t}\, dS = \int_{s} (\ell u + mv)\left(\rho\frac{q^2}{2} + \rho\,\Omega + p\right) ds$$

$$ds^2 = dx^2 + dy^2 \qquad q^2 = u^2 + v^2 \qquad dS = dx\,dy$$

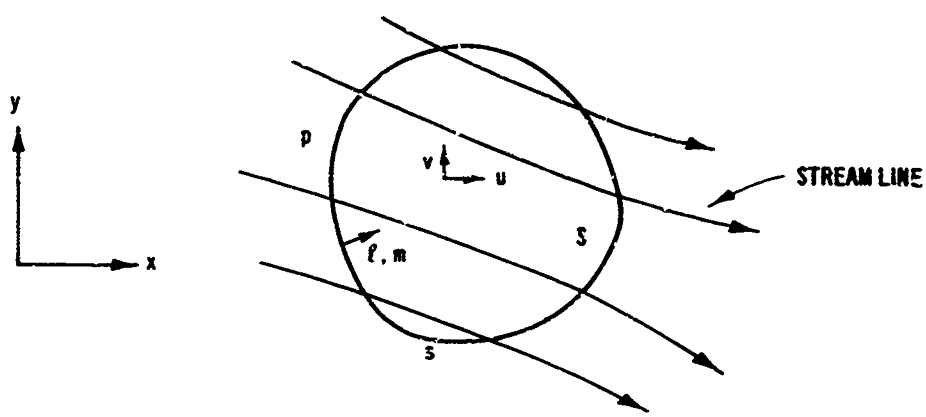


Fig. 2. Two forms of energy equation

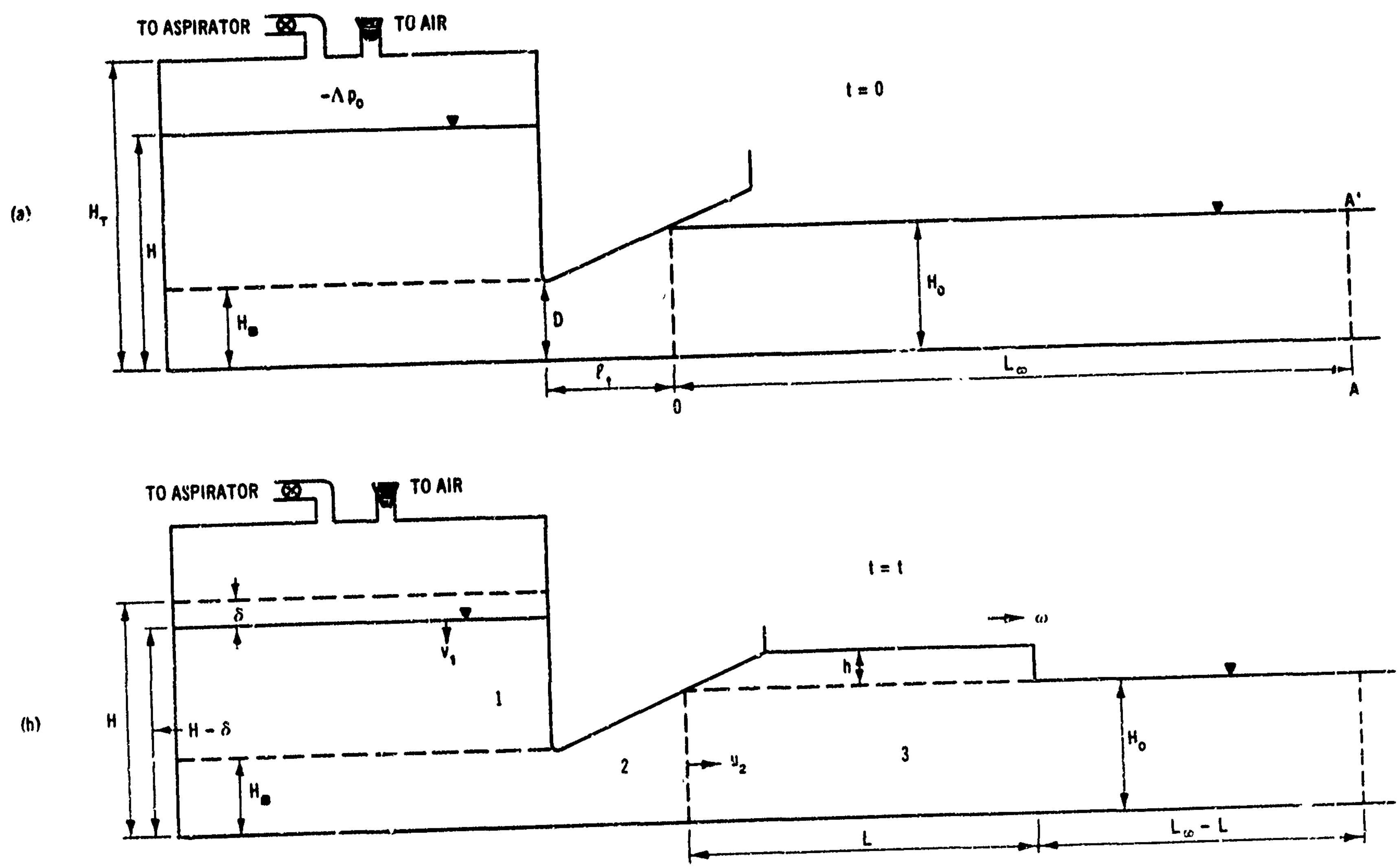


Fig. 3. Initial and subsequent dispositions of water surfaces; shallow-water long wave

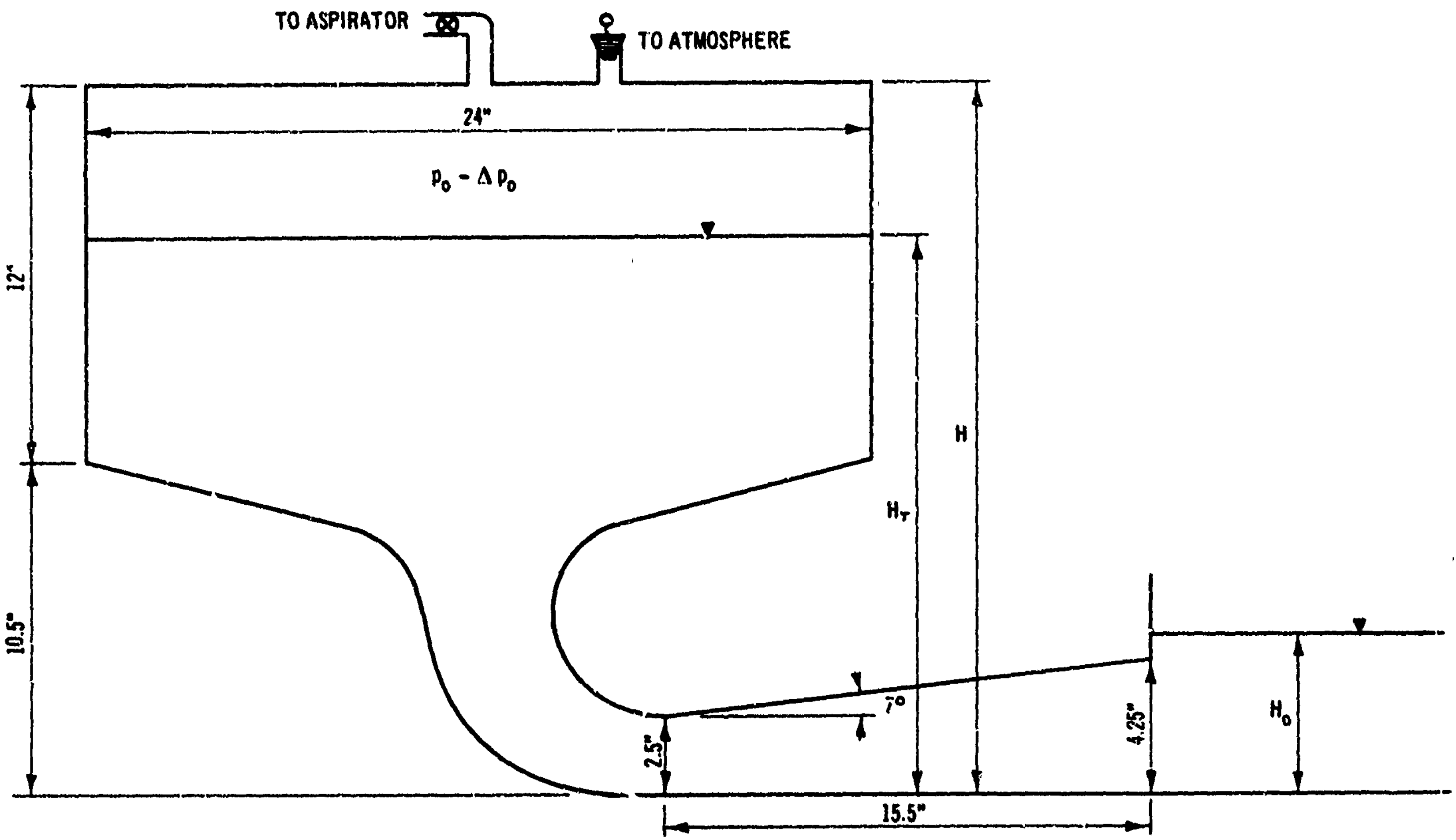


Fig. 4. Elevated pneumatic chamber

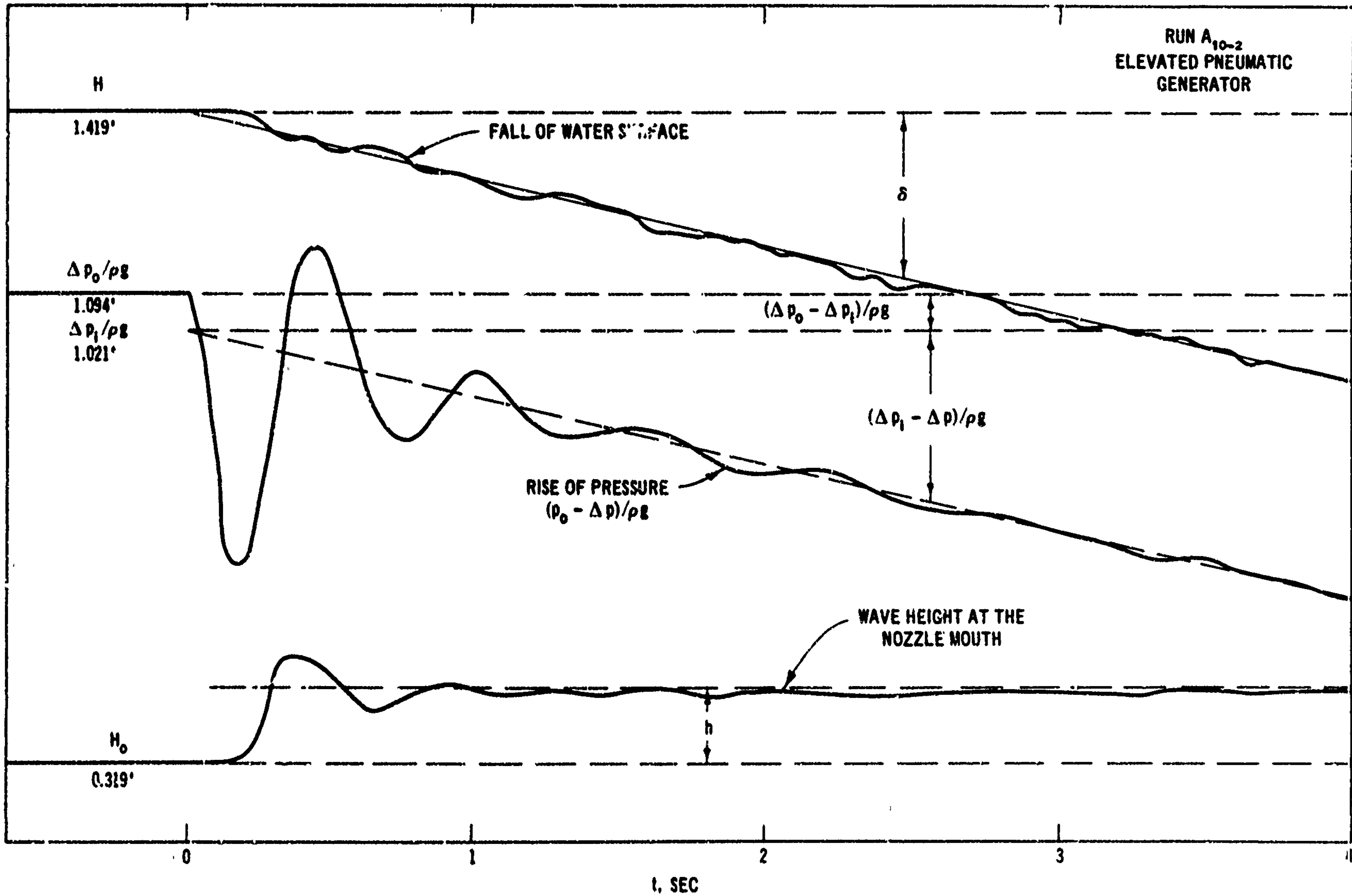


Fig. 5. Record of chamber water elevation, pressure, and wave height

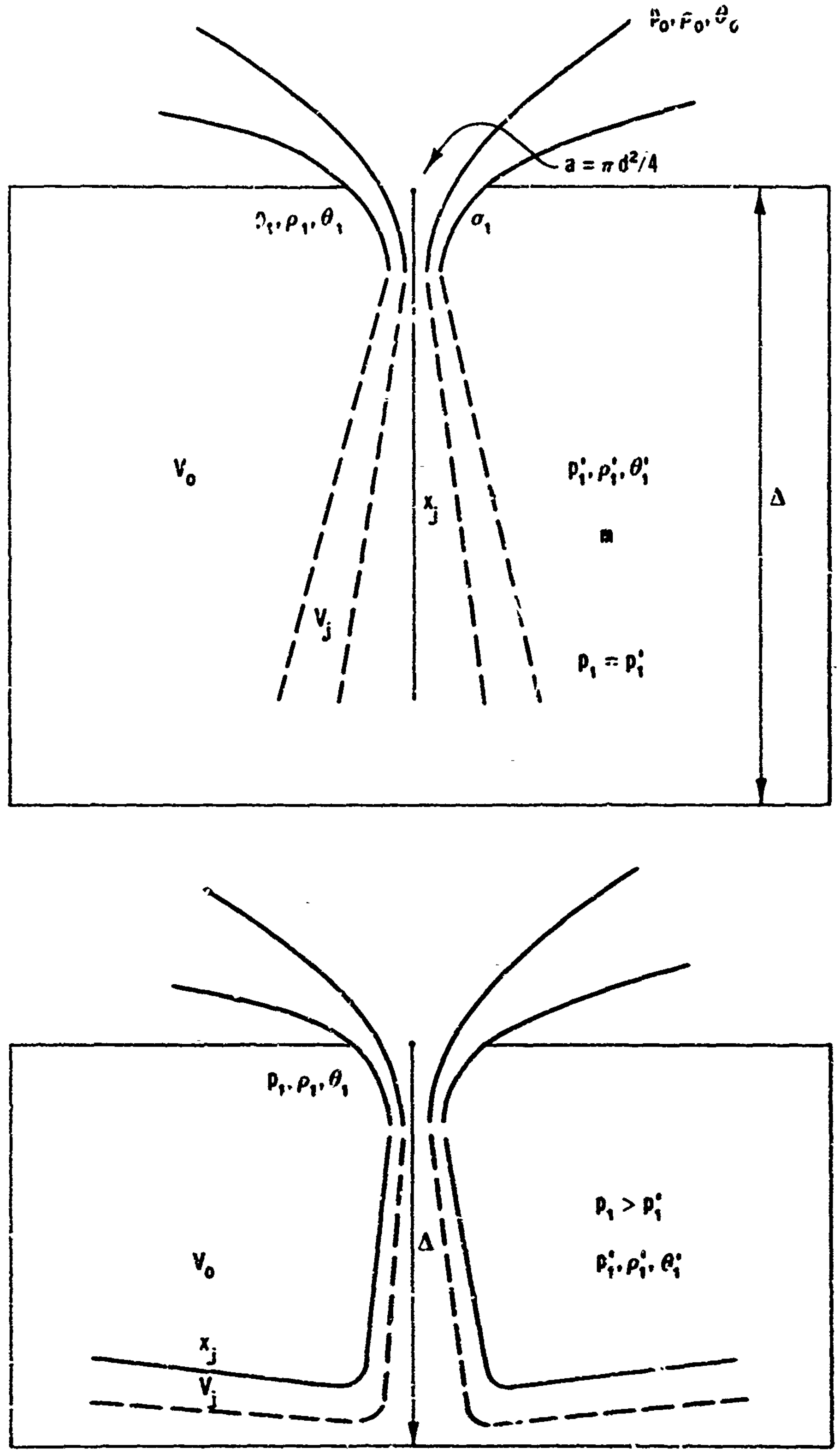


Fig. 6. Notation diagram. Air flow into
a vessel of finite volume

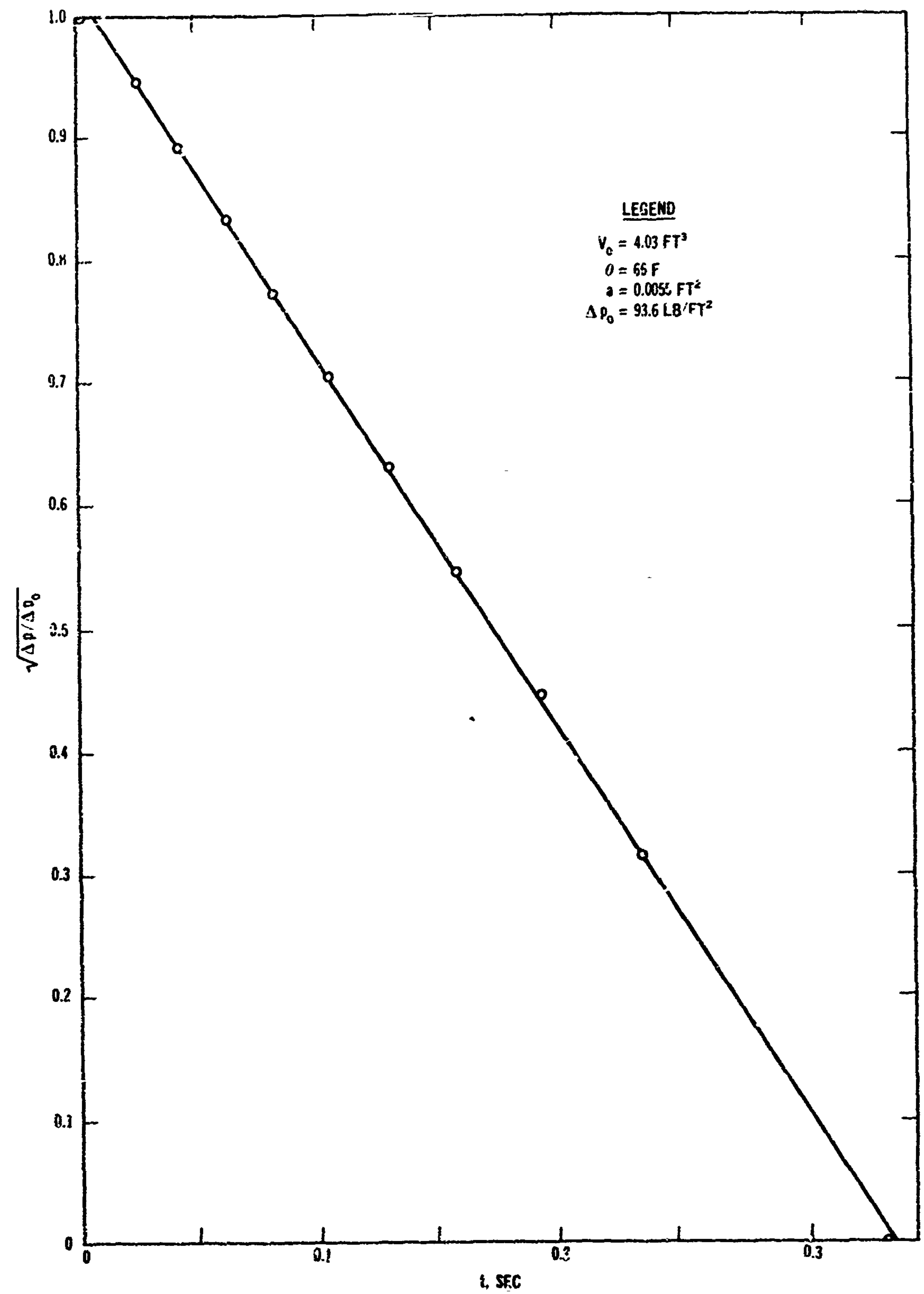


Fig. 1. Increase of pressure with time in filling vessel

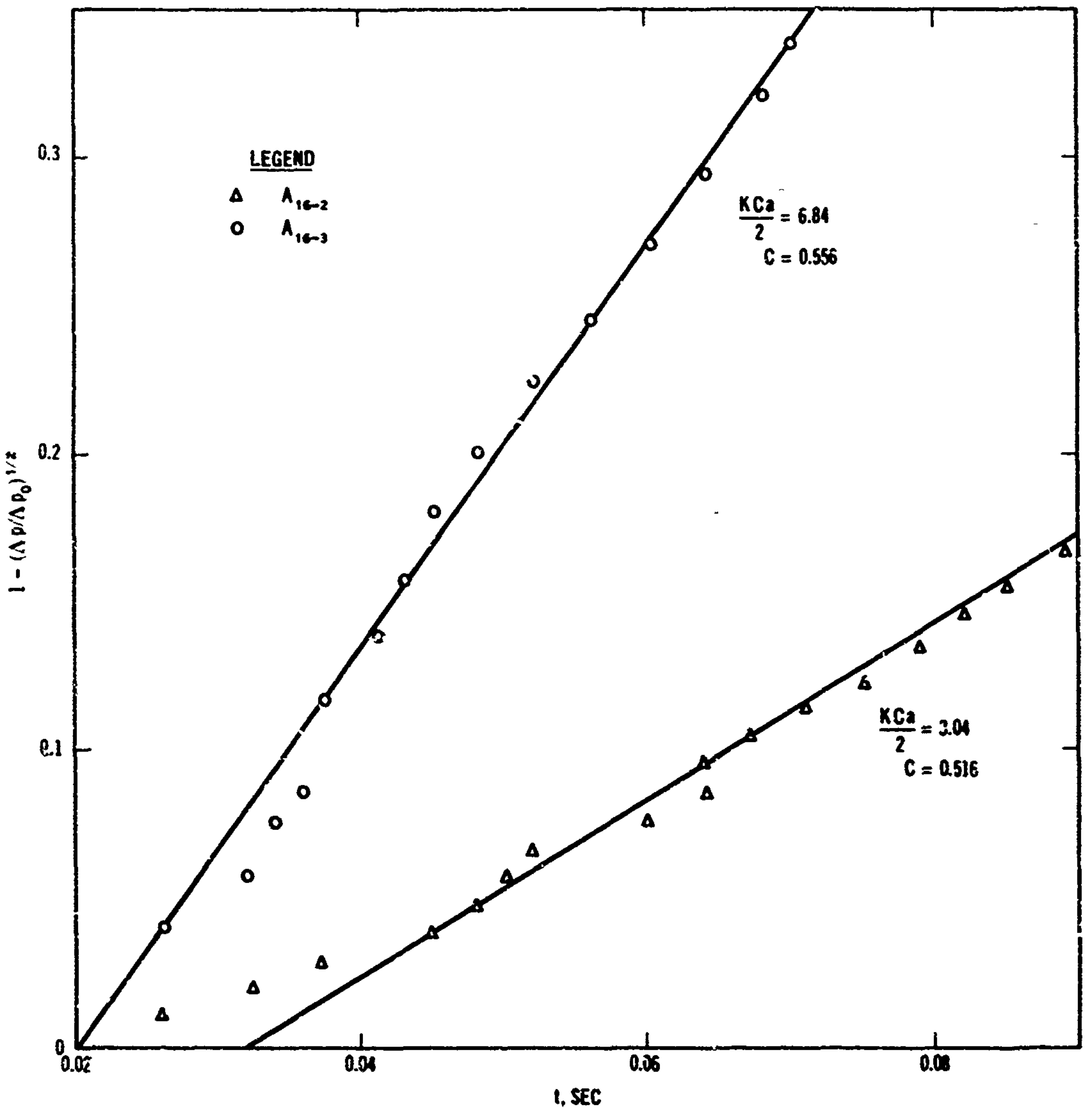


Fig. 8. Initial rise of pressure in elevated pneumatic chamber; free orifice

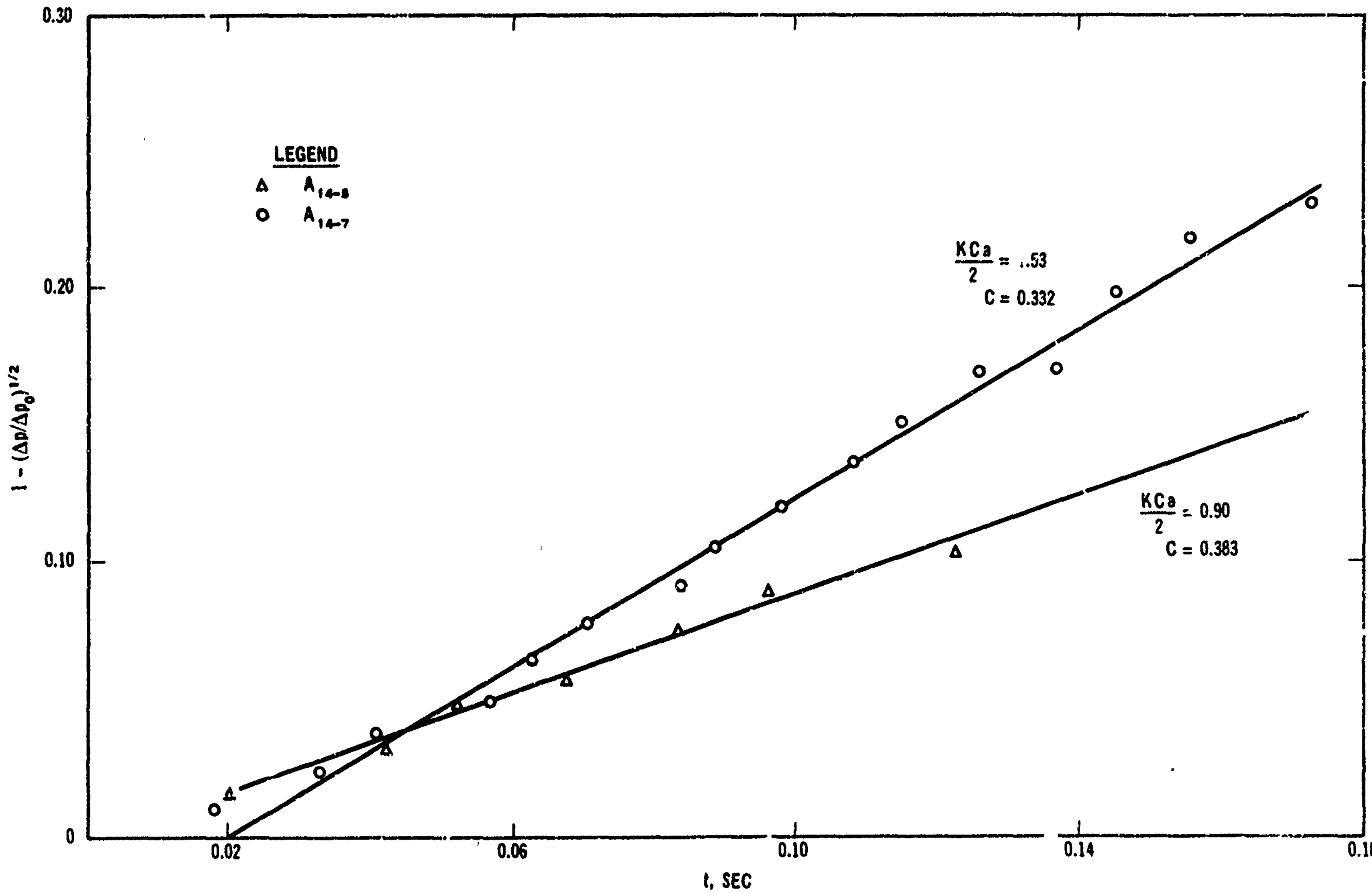


Fig. 9. Initial rise of pressure in a low pneumatic chamber; free orifice

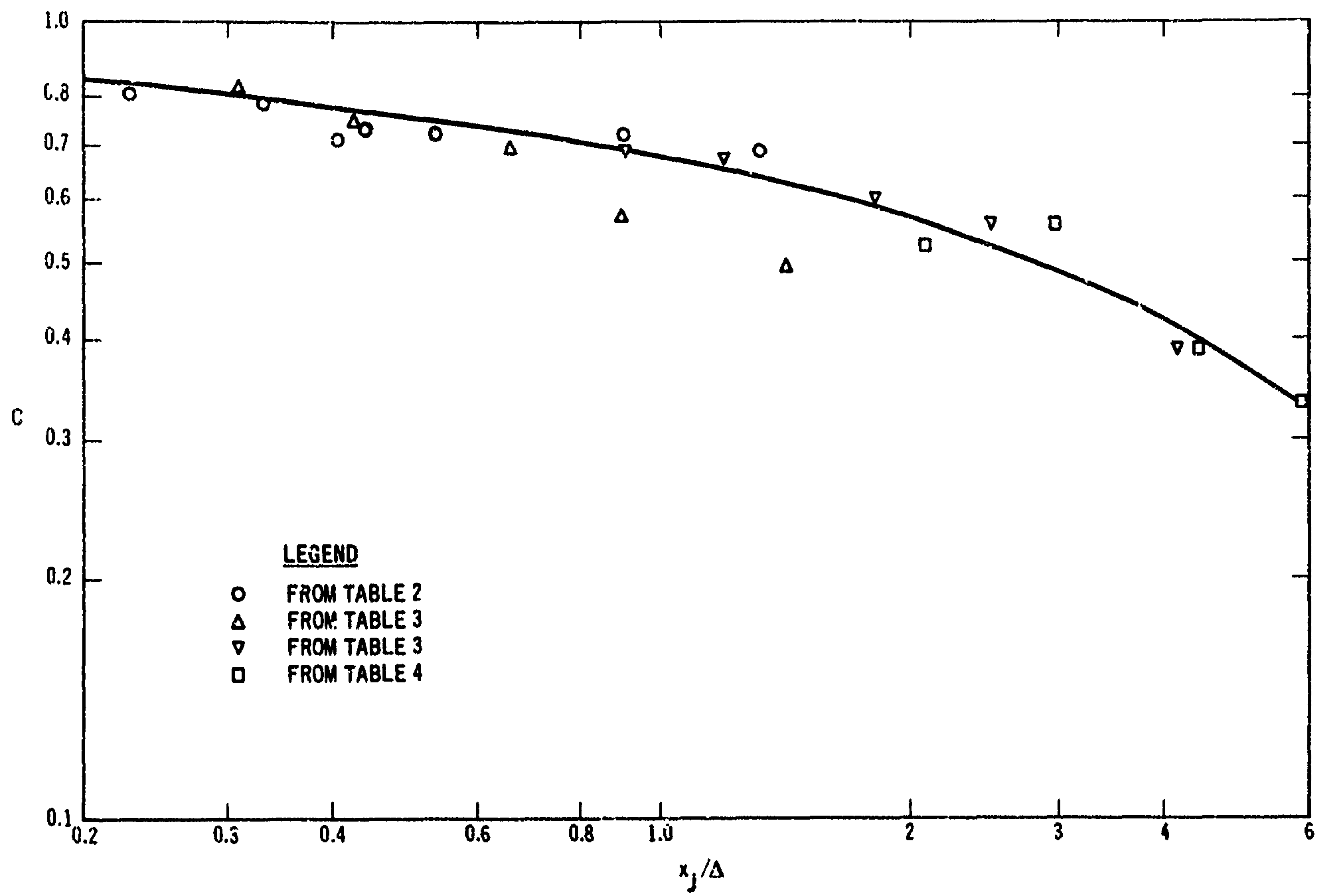


Fig. 10. Discharge coefficient as function of vessel depth

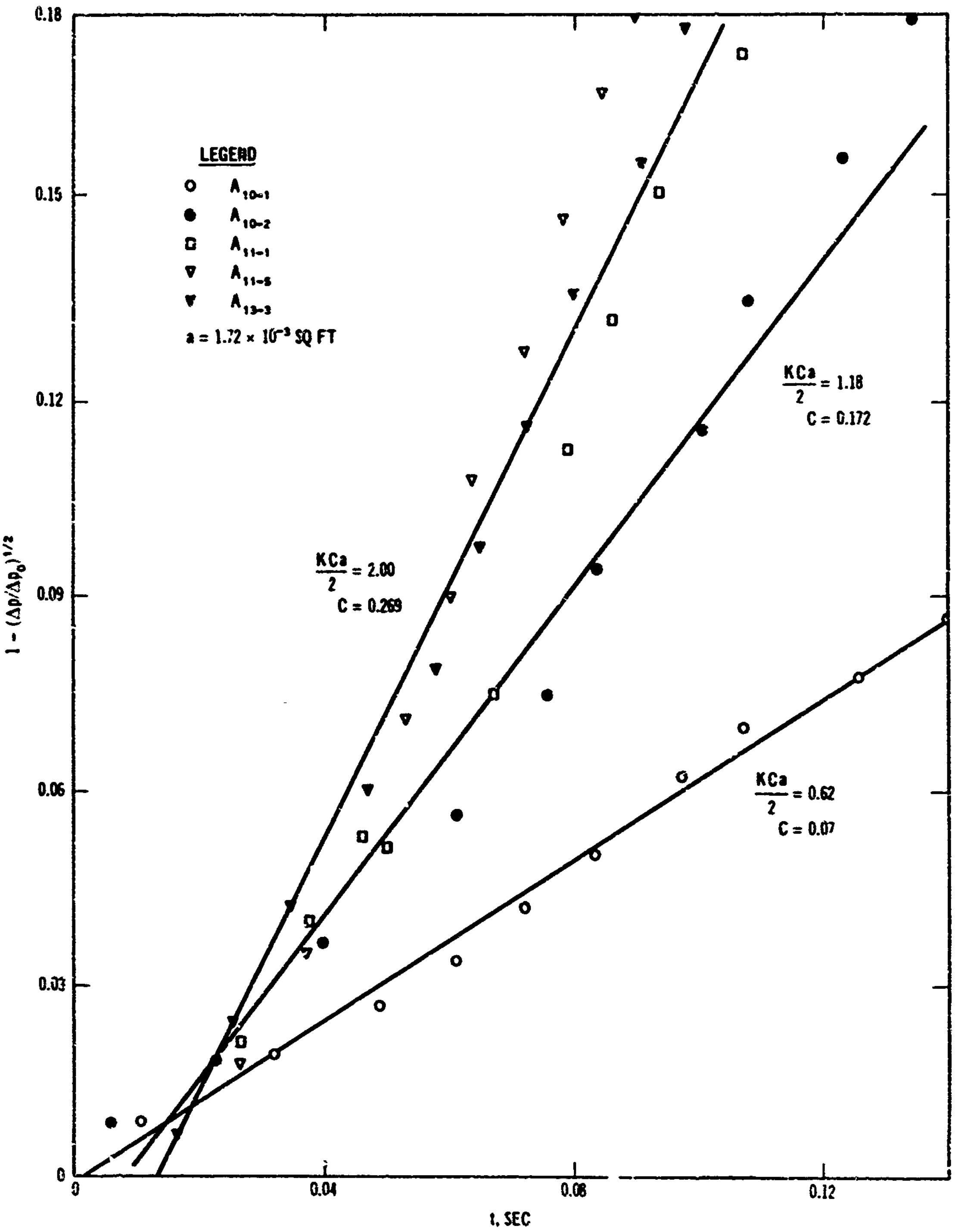


Fig. 11. Initial rise of pressure in a pneumatic chamber operated with a three-way valve

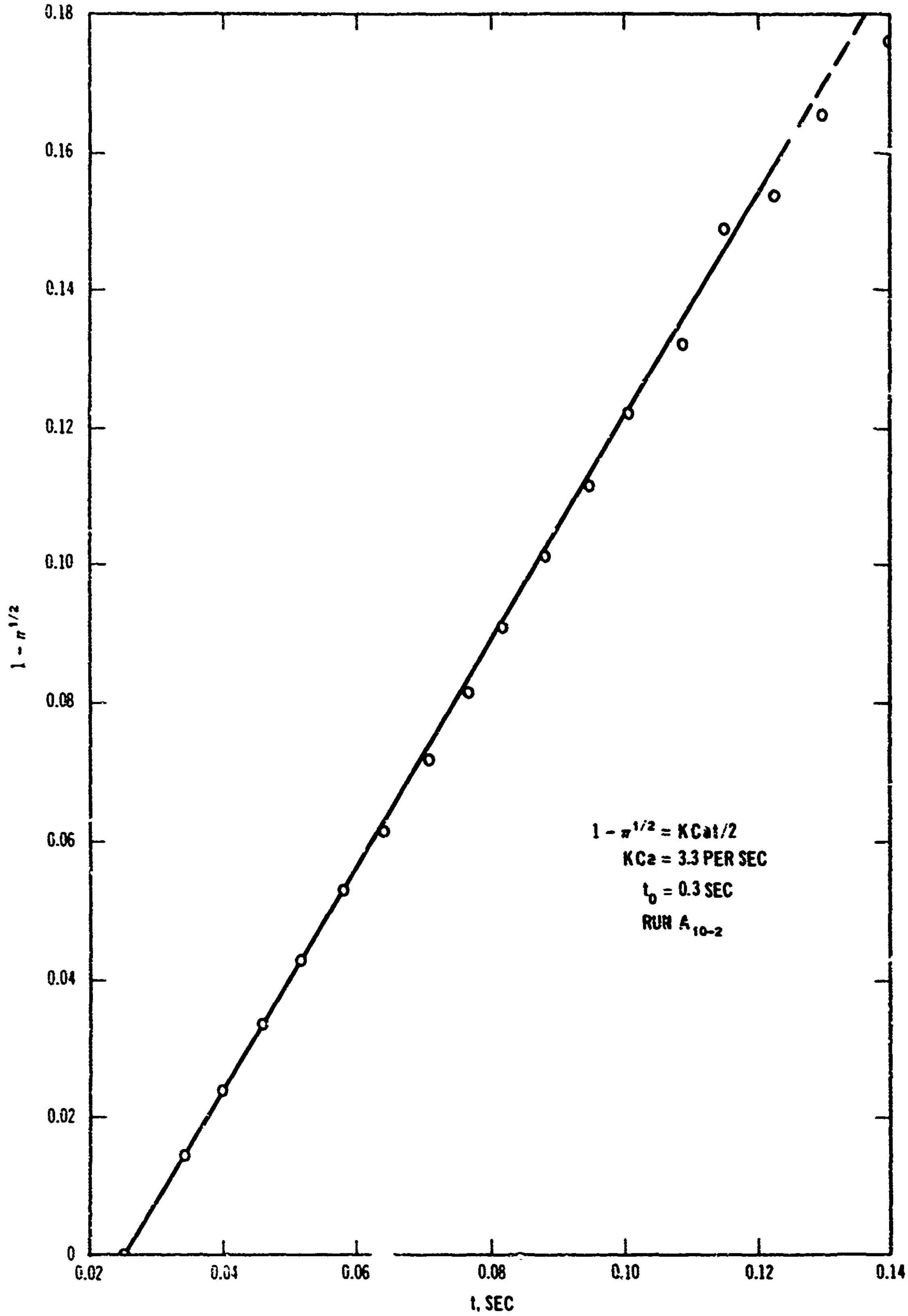


Fig. 12. Initial pressure increase, water in chamber still

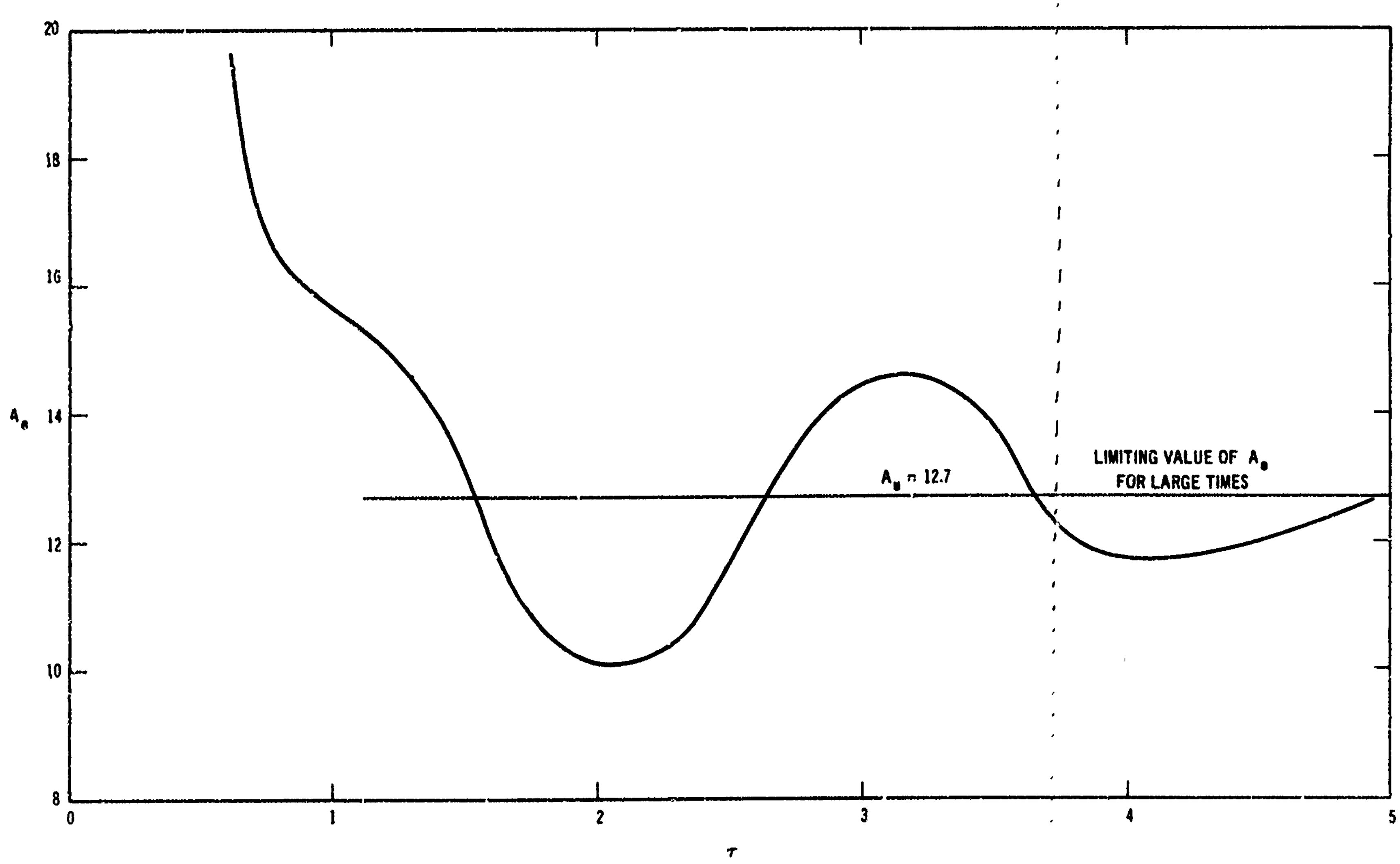


Fig. 13. Variation of A_8 of air flow equation with time

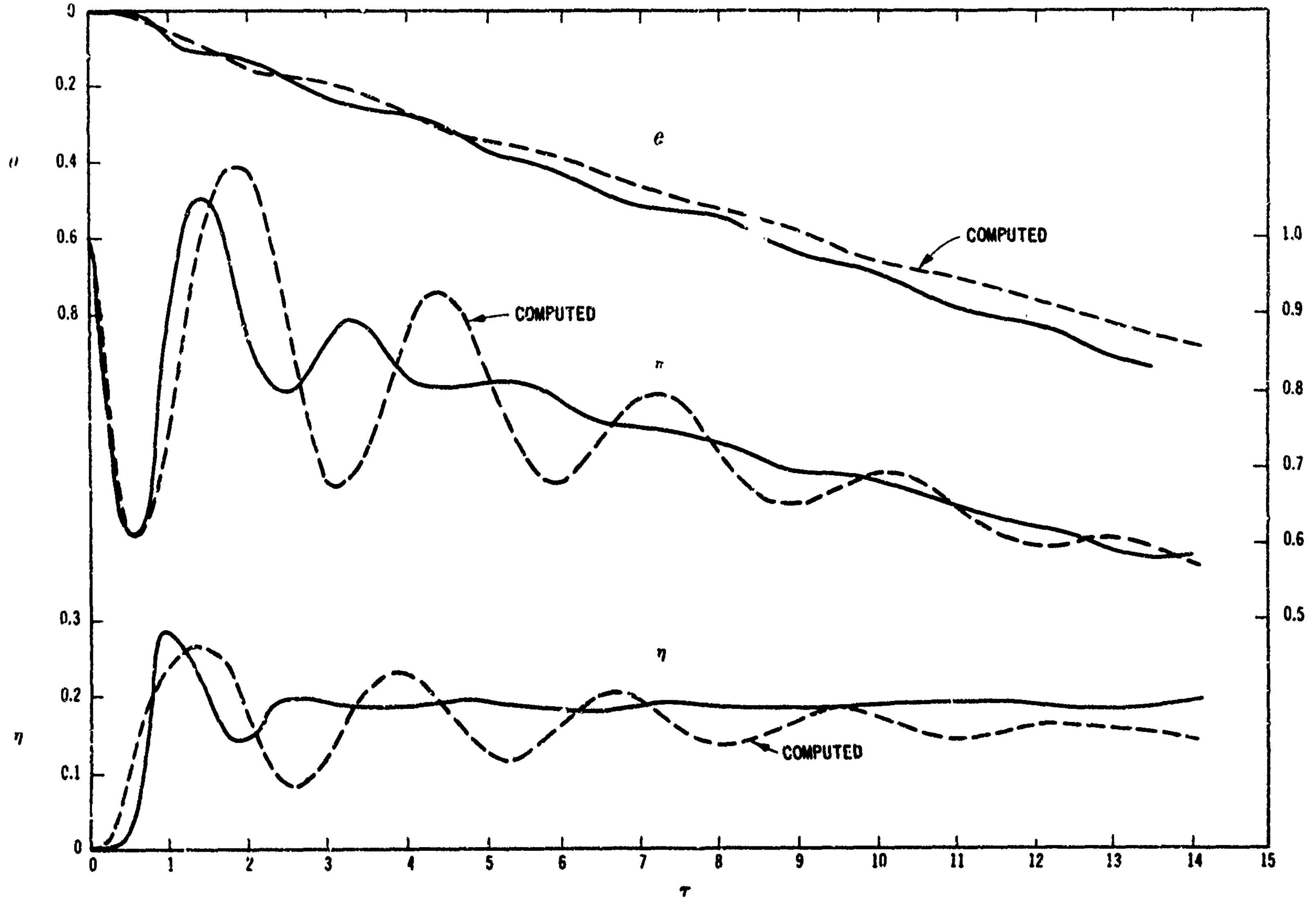


Fig. 14. Comparison between observed and computed quantities, run A_{10-2}

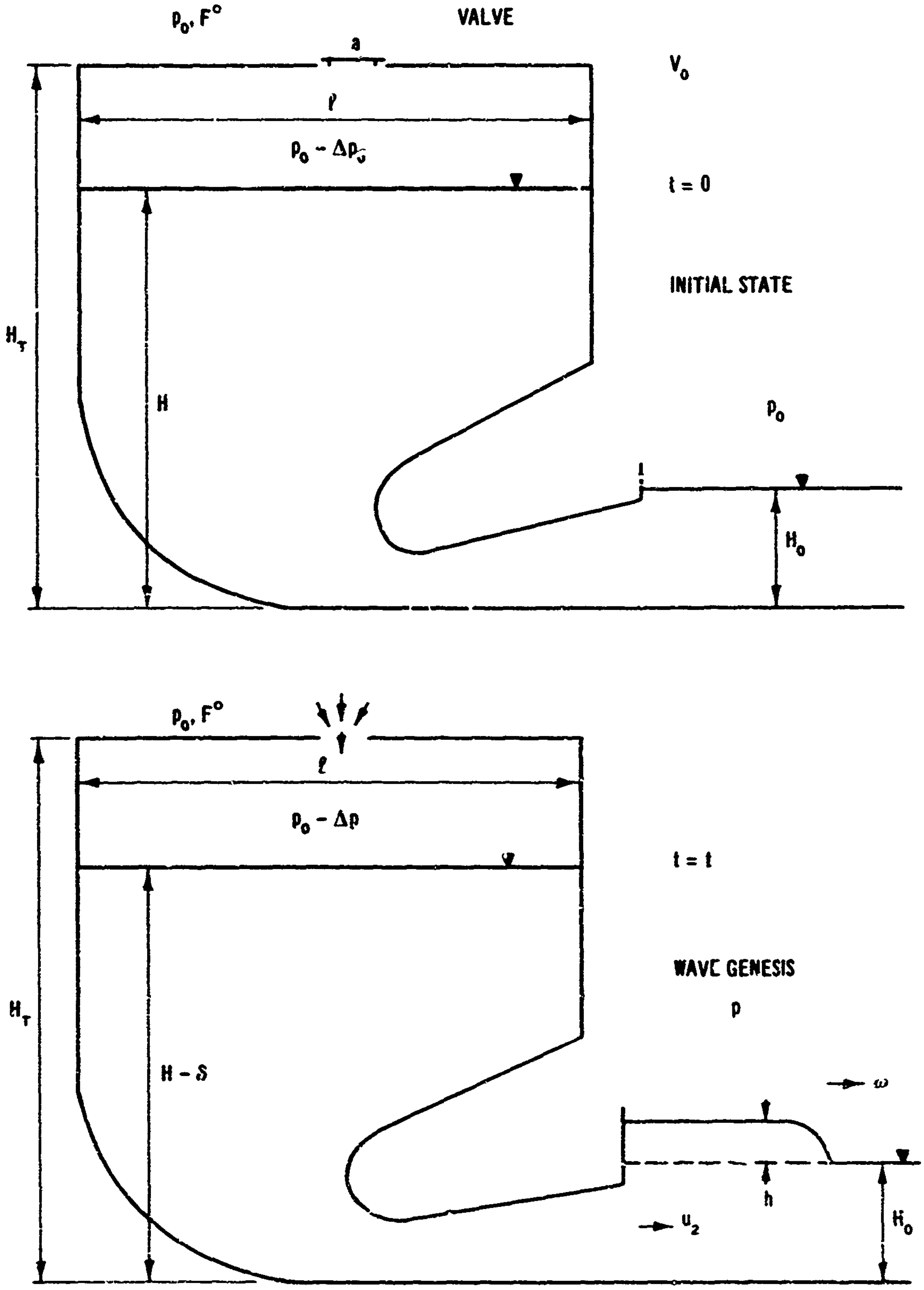


Fig. 15. Notation diagram. Elevated pneumatic generators

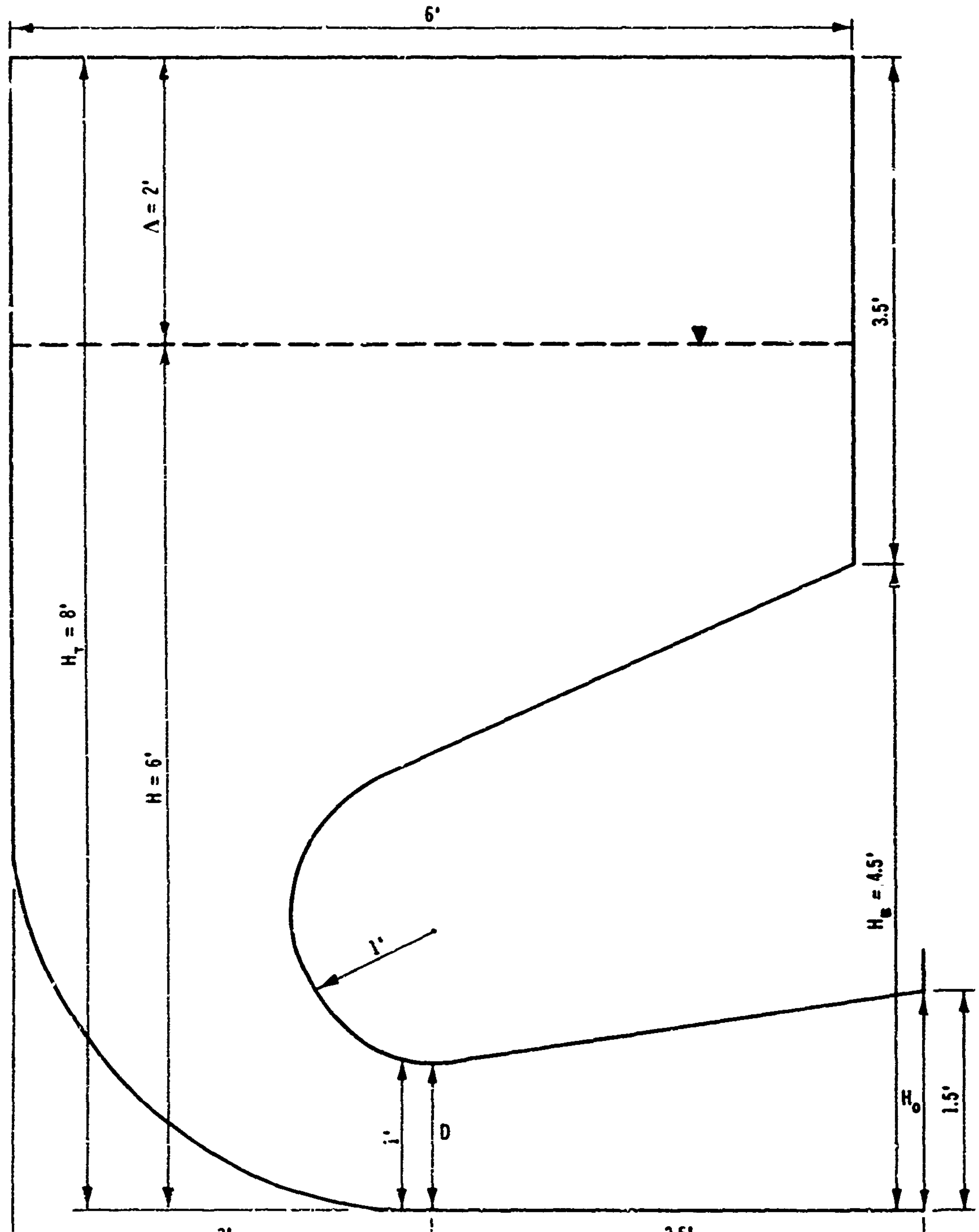


Fig. 1(. Pneumatic generator for Hilo Bay tsunami model

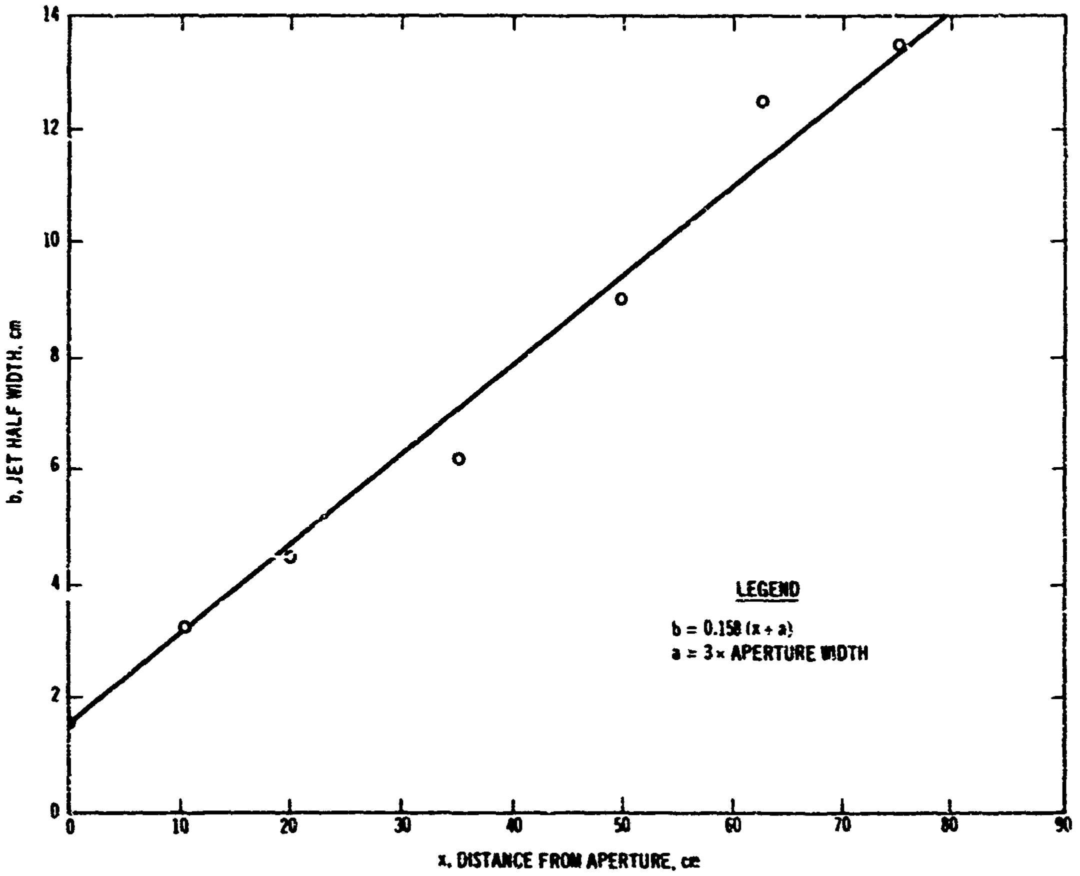


Fig. 17. Turbulent expansion after Förthman

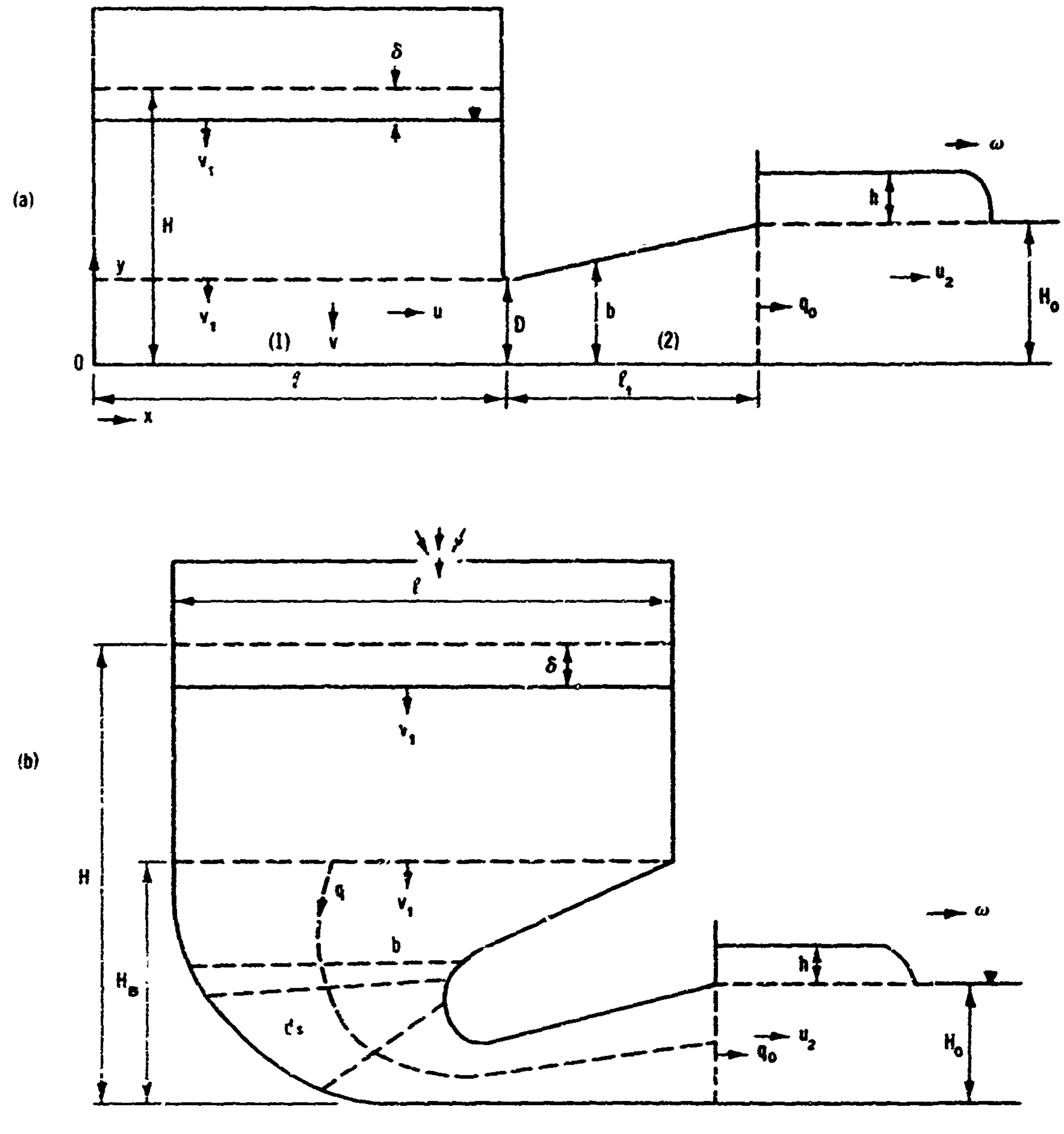


Fig. 18. Notation diagram relative to kinetic energy in chamber and nozzle liquid

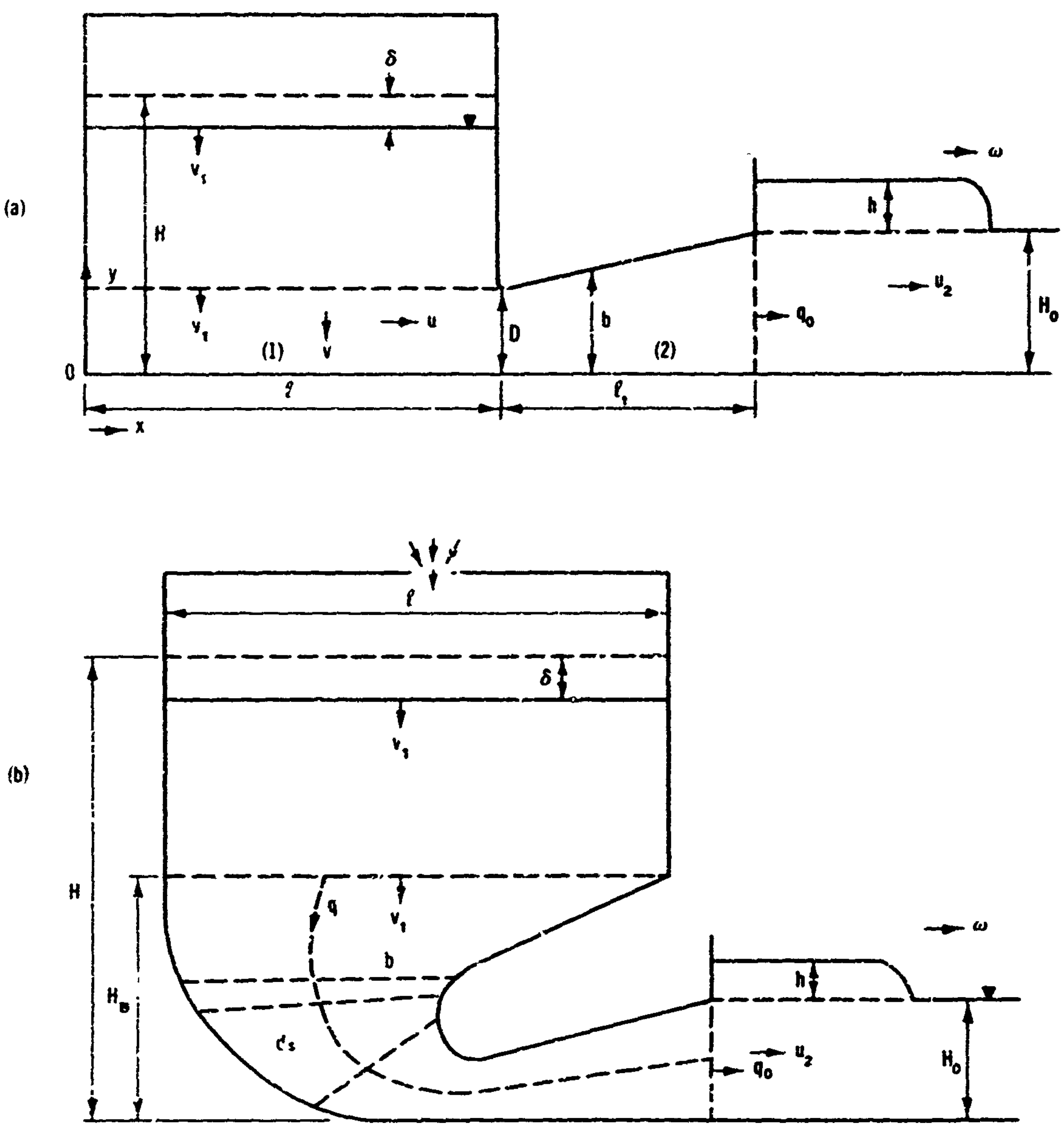


Fig. 18. Notation diagram relative to kinetic energy in chamber and nozzle liquid